Math 4250/6250

Key ideas of course:

Math started with geometry.

Shapes are all around you — but they aren't accidental. <u>Why</u>?

Linear algebra, differential equations and calculus are surprisingly powerful tools for understanding shape.

Definition. The <u>dot product</u> of vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is given by

$$\langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i$$

Definition. The length (or norm) of a vector $\vec{v} \in \mathbb{R}^n$ is given by

$$\| \vec{v} \| = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

Theorem (Cauchy-Schwartz) For any $\vec{v}, \vec{w} \in \mathbb{R}^n$, $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|$ with equality $\Leftrightarrow$ one vector is a scalar multiple of the other.

Definition. The angle between $\vec{v}, \vec{w} \in \mathbb{R}^n$ is defined by $\cos\theta = \dfrac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|}$, $\theta \in [0, \pi]$.

Proof (of C-S). If $\vec{w} = \vec{0}$, we're done.

If $\vec{w} \neq \vec{0}$, consider

$$g(t) = \|\vec{x} + t\vec{y}\|^2$$

$$= \langle \vec{x} + t\vec{y}, \vec{x} + t\vec{y} \rangle$$

$$= \langle \vec{x}, \vec{x} \rangle + 2t\langle \vec{x}, \vec{y} \rangle + t^2 \langle \vec{y}, \vec{y} \rangle$$

We can find the minimum of this quadratic function of t by differentiating

$$g'(t) = 2\langle \vec{x}, \vec{y} \rangle + 2t\langle \vec{y}, \vec{y} \rangle$$

and solving $g'(t) = 0$ to get

$$t_0 = -\frac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}$$

Now $g(t) \geq 0$ for all t, so we

Know that

$$g(t_0) = \langle \vec{x}, \vec{x} \rangle - 2\frac{\langle \vec{x}, \vec{y} \rangle^2}{\langle \vec{y}, \vec{y} \rangle} + \frac{\langle \vec{x}, \vec{y} \rangle^2}{\langle \vec{y}, \vec{y} \rangle^{\cancel{2}}} \cancel{\langle \vec{y}, \vec{y} \rangle}$$

$$= \langle \vec{x}, \vec{x} \rangle - \frac{\langle \vec{x}, \vec{y} \rangle^2}{\langle \vec{y}, \vec{y} \rangle} \geq 0$$

Rearranging, we get

$$\langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle \geq \langle \vec{x}, \vec{y} \rangle^2$$

or (taking square roots)

$$\|\vec{x}\| \, \|\vec{y}\| \geq |\langle \vec{x}, \vec{y} \rangle|.$$

We have equality $\Leftrightarrow$ $g(t_0) = 0$, in which case $\vec{x} = -t_0 \vec{y}$ and $\vec{x}$ is a scalar multiple of $\vec{y}$. $\blacksquare$

We are going to use this to prove

Proposition. (The triangle inequality)
For any vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

with equality $\iff$ $\vec{v}$ is a positive scalar multiple of $\vec{w}$.

Proof. We compute

$$\begin{aligned}
\|\vec{v}+\vec{w}\|^2 &= \langle \vec{v}+\vec{w}, \vec{v}+\vec{w} \rangle \\
&= \langle \vec{v}, \vec{v} \rangle + 2\langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{w} \rangle \\
&\leq \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2 \\
&= (\|\vec{v}\| + \|\vec{w}\|)^2. \qquad \blacksquare
\end{aligned}$$

Cauchy Schwartz!

Now we switch from "geometry" to "differential geometry" by proving an integrated version of this inequality.

Theorem. If $\vec{\alpha}(t) : [a,b] \to \mathbb{R}^n$ is a continuous vector-valued function, then

$$\left\| \int_a^b \vec{\alpha}(t)\, dt \right\| \leq \int_a^b \|\alpha(t)\|\, dt.$$

with equality $\iff \dfrac{\vec{\alpha}(t)}{\|\vec{\alpha}(t)\|}$ is constant.

Proof. Let $\vec{v} \in \mathbb{R}^n$ be any vector with $\|\vec{v}\| = 1$. Then for each $t \in [a,b]$,

$$\|\vec{\alpha}(t)\| = \|\vec{\alpha}(t)\|\,\|\vec{v}\|$$

Cauchy-Schwartz $\rightarrow$ $\geq \langle \vec{\alpha}(t), \vec{v} \rangle$

$$\int_a^b \|\vec{\alpha}(t)\|\, dt \geq \int_a^b \langle \vec{\alpha}(t), \vec{v} \rangle\, dt$$

ith component of $\vec{\alpha}(t)$

$$= \int_a^b \sum_{i=1}^n \alpha_i(t)\, v_i\, dt$$

$$= \sum_{i=1}^{n} v_i \int_a^b \alpha_i(t)\,dt$$

$$= \left\langle \vec{v}, \int_a^b \vec{\alpha}(t)\,dt \right\rangle$$

Now this is true for all $\vec{v}$, so wlog we may assume that $\vec{v}$ is a scalar multiple of $\int_a^b \vec{\alpha}(t)\,dt$. By Cauchy-Schwartz, in that case,

$$= \|\vec{v}\| \left\| \int_a^b \vec{\alpha}(t)\,dt \right\|$$

$$= \left\| \int_a^b \vec{\alpha}(t)\,dt \right\|.$$

Equality holds if $\vec{\alpha}(t)$ is a positive scalar multiple of $\vec{v}$ for each t, or $\frac{\vec{\alpha}(t)}{\|\vec{\alpha}(t)\|} = \vec{v}$ for all t. $\blacksquare$

We start by studying curves.

Definition. A function $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ is called a parametrized curve. We write $\vec{\alpha}(t) = (\alpha_1(t), \ldots, \alpha_n(t))$.

Recall that the derivative of $\vec{\alpha}$,

$$\vec{\alpha}'(t) = (\alpha_1'(t), \ldots, \alpha_n'(t))$$

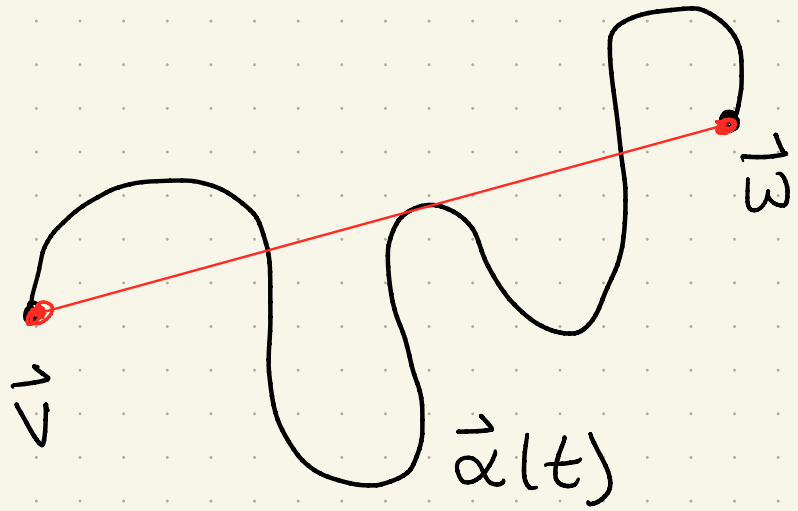
is also a vector valued function.

Definition. The length $\|\vec{\alpha}'(t)\|$ is called the <u>speed</u> of $\vec{\alpha}(t)$. The vector $\vec{\alpha}'(t)$ is called the <u>velocity vector</u> of $\vec{\alpha}(t)$.

Definition. If $\vec{\alpha}: [a,b] \to \mathbb{R}^n$, the <u>length of the curve</u> is

$$\int_a^b \|\vec{\alpha}'(t)\| \, dt$$

We are now ready to prove our first main theorem!



Theorem. If $\vec{v}, \vec{w} \in \mathbb{R}^n$ are any two points, a shortest differentiable curve $\vec{\alpha} : [0,1] \to \mathbb{R}^n$ with $\vec{\alpha}(0) = \vec{v}$ and $\vec{\alpha}(1) = \vec{w}$ is the straight line

$$\vec{\alpha}(t) = \vec{v} + t(\vec{w} - \vec{v})$$

which has length $\|\vec{v} - \vec{w}\|$.

Proof. If $\vec{\alpha}:[0,1]\to\mathbb{R}^n$ is diff.,

$$\vec{\alpha}(1)-\vec{\alpha}(0)=\int_0^1 \vec{\alpha}'(t)\,dt$$

and so

$$\|\vec{\alpha}(1)-\vec{\alpha}(0)\| = \left\|\int_0^1 \vec{\alpha}'(t)\,dt\right\|$$

$$\leq \int_0^1 \|\vec{\alpha}'(t)\|\,dt = \text{length of } \vec{\alpha}.$$

Thus the length of $\vec{\alpha}$ is at least $\|\vec{\alpha}(1)-\vec{\alpha}(0)\| = \|\vec{w}-\vec{v}\|$. If

$$\vec{\alpha}(t)=\vec{v}+t(\vec{w}-\vec{v})$$

then $\vec{\alpha}'(t)=\vec{w}-\vec{v}$, and $\|\vec{\alpha}'(t)\|=\|\vec{w}-\vec{v}\|$, so the length is exactly $\int_0^1 \|\vec{w}-\vec{v}\|\,dt = \|\vec{w}-\vec{v}\|$. $\boxtimes$

Notice that we <u>haven't</u> proved that the line is the <u>unique</u> length minimizing curve. (That will take more theory!).

Now suppose we have <u>two</u> curves

Proposition. If $\vec{\alpha}, \vec{\beta} : \mathbb{R} \to \mathbb{R}^n$ are differentiable functions, then $\langle \vec{\alpha}(t), \vec{\beta}(t) \rangle$ is differentiable and

$$\frac{d}{dt} \langle \vec{\alpha}(t), \vec{\beta}(t) \rangle = \langle \vec{\alpha}'(t), \vec{\beta}(t) \rangle + \langle \vec{\alpha}(t), \vec{\beta}'(t) \rangle.$$

Proof. (homework)

Proposition. If $\vec{\alpha}(t)$ is a vector valued function so that $\|\vec{\alpha}(t)\| \equiv 1$, then $\langle \vec{\alpha}'(t), \vec{\alpha}(t) \rangle = 0$.

Proof. If $\|\vec{\alpha}(t)\| \equiv 1$, then $\|\vec{\alpha}(t)\|^2 \equiv 1$, or $\langle \vec{\alpha}(t), \vec{\alpha}(t) \rangle \equiv 1$. Differentiating, both sides,

$$0 = \frac{d}{dt} \langle \vec{\alpha}(t), \vec{\alpha}(t) \rangle = \langle \vec{\alpha}'(t), \vec{\alpha}(t) \rangle + \langle \vec{\alpha}(t), \vec{\alpha}'(t) \rangle$$
$$= 2 \langle \vec{\alpha}'(t), \vec{\alpha}(t) \rangle. \quad \boxtimes$$

We now recall (for vectors in $\mathbb{R}^3$)

Definition. If $\vec{v}, \vec{w} \in \mathbb{R}^3$, the <u>cross product</u>

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

Properties.

1) The cross product is bilinear.
2) $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$
3) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$,
   where θ is the angle between $\vec{v}$ and $\vec{w}$.
4) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

We will often use

Definition. If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, the <u>triple product</u> is $\langle \vec{u}, \vec{v} \times \vec{w} \rangle$.

Properties.

1) The triple product is trilinear.
2) $\langle \vec{u}, \vec{v} \times \vec{w} \rangle = \det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$
3) $\langle \vec{u}, \vec{v} \times \vec{w} \rangle = -\langle \vec{v}, \vec{u} \times \vec{w} \rangle = -\langle \vec{u}, \vec{w} \times \vec{v} \rangle$
   $= -\langle \vec{w}, \vec{v} \times \vec{u} \rangle$

Proposition. If $\vec{\alpha}(t)$, $\vec{\beta}(t)$ are vector valued functions in $\mathbb{R}^3$,

$$\frac{d}{dt}\vec{\alpha}(t)\times\vec{\beta}(t) = \vec{\alpha}'(t)\times\vec{\beta}(t) + \vec{\alpha}(t)\times\vec{\beta}'(t).$$

Proof. (Homework)

note the order

This gives us our first set of tools to think about curve geometry.

Next time, we'll practice constructing some curves to play with.