

Differential Forms

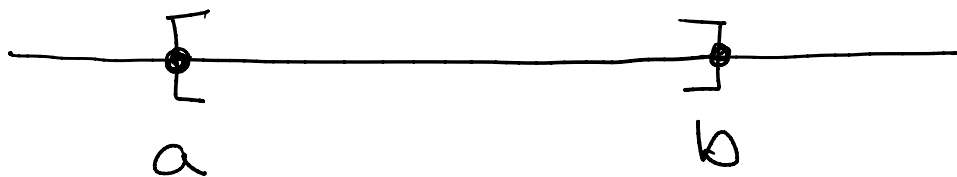
8.1/8.2 Differential Forms.

We now understand how to differentiate and integrate in multivariable calculus.

We want to connect the two operations by proving the fundamental theorem of calculus. In one variable, this is

$$\int_a^b f(x) dx = F(b) - F(a)$$

whenever $f(x) = F'(x)$. Now



So we can observe: (in our new language)

$[a, b]$ is a region Ω

$[a, b]$ is the closure of (a, b)

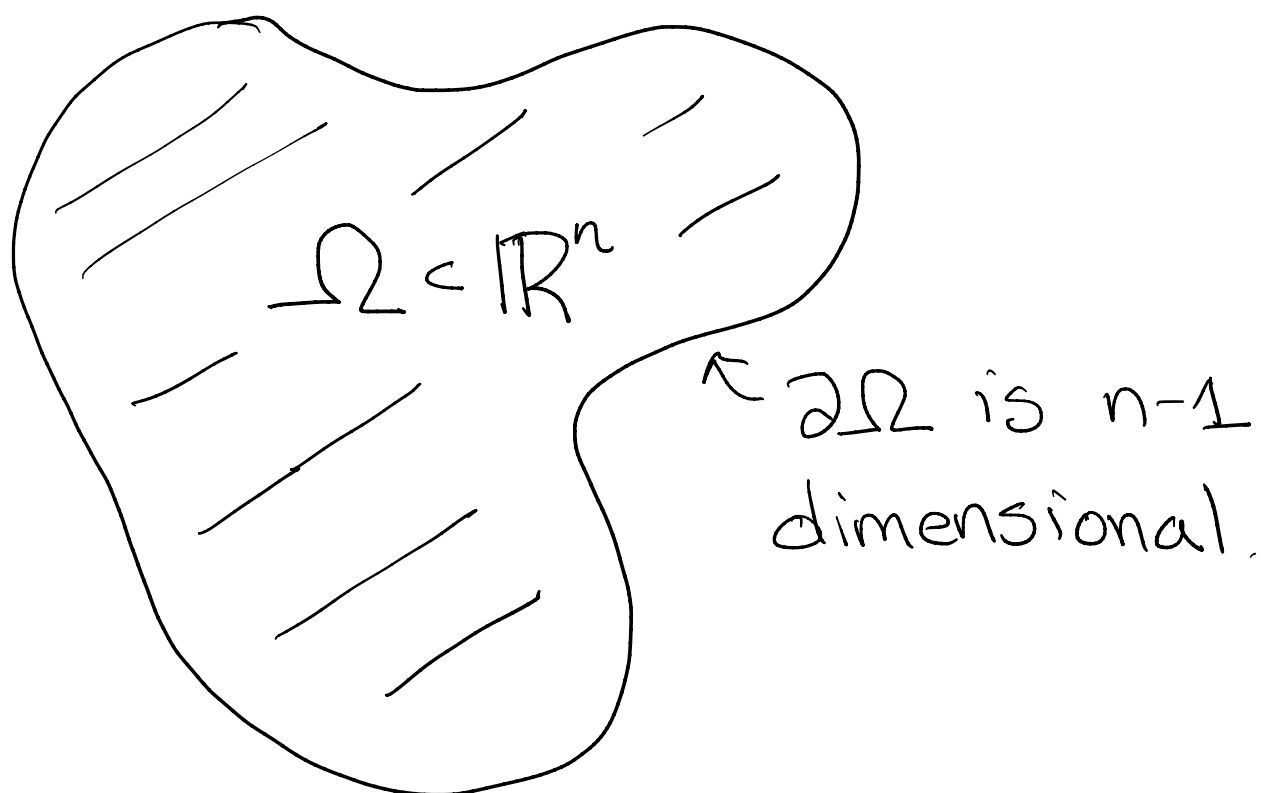
$\{a, b\} = \text{bdy}(\Omega) = \partial\Omega$ is the set of frontier or boundary points of Ω .

So the theorem has the structure

$$\int_{\Omega} dF(x) \, dx = \int_{\{b\}} F(x) \, dx - \int_{\{a\}} F(x) \, dx$$

if we realize that "0-dimensional integration" is just evaluation at the point.

To extend this to higher dimensional regions, we'll need to keep track of several things:



F something that can be integrated over $(n-1)$ -dimensional manifolds

dF something that can be integrated over n -dimensional manifold

and our goal will be to prove
Stokes's Theorem:

$$\int_{\Omega} dF = \int_{\partial\Omega} F$$

So what are F and dF ?

They are a special kind of
vector-valued function called
a differential form.

Defining differential forms will
take a few classes.

Definition. The vector space of linear maps $\mathbb{R}^n \rightarrow \mathbb{R}$ is called the dual space $(\mathbb{R}^n)^*$.

The dual space is an n -dimensional vector space. If \mathbb{R}^n has the standard basis $\vec{e}_1, \dots, \vec{e}_n$ then (\mathbb{R}^n) has a standard dual basis dx_1, \dots, dx_n defined by

$$dx_i(\vec{e}_j) = \delta_{ij}$$

so

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow dx_i(\vec{v}) = v_i$$

Homework: Prove that $\{dx_i\}$ is a basis for $(\mathbb{R}^n)^*$.

Now we can extend this idea:

Definition. The vector space of alternating multilinear functions $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k \text{ times}} \rightarrow \mathbb{R}$ is denoted $\Lambda^k(\mathbb{R}^n)$.

We define $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$.

We know that \uparrow functions with no inputs are constant

$\Lambda^n(\mathbb{R}^n)$ is one-dim. (det)

We call this space "alt-n-k"

or "Lambda-n-k".

and

$$\Lambda^1(\mathbb{R}^n) = (\mathbb{R}^n)^* = n\text{-dimensional}$$

Now we want to build a basis for the "middle guys" $\Lambda^k(\mathbb{R}^n)$.

Definition. A k -index I in \mathbb{R}^n is an ordered (i_1, \dots, i_k) with each $i_j \in \{1, \dots, n\}$.

Note: When we say "ordered", we mean that the order of the i_j matters: $(7, 3, 7)$ and $(3, 7, 7)$ are different 3 indices in \mathbb{R} . There's also no requirement that the indices i_j are all different.

Given a k -index I in \mathbb{R}^n ,
we define

$$dx_I(\vec{v}_1, \dots, \vec{v}_k) = \det V_I$$

$$= \det \begin{bmatrix} dx_{i_1}(\vec{v}_1) & \dots & dx_{i_1}(\vec{v}_k) \\ \vdots & \ddots & \vdots \\ dx_{i_k}(\vec{v}_1) & \dots & dx_{i_k}(\vec{v}_k) \end{bmatrix}$$

where V is the $n \times k$ matrix

$$V = \begin{bmatrix} \uparrow \vec{v}_1 & \dots & \vec{v}_k \uparrow \\ \downarrow & & \downarrow \end{bmatrix} \text{ and } V_I \text{ is the}$$

submatrix formed by taking

rows i_1, \dots, i_k .

Example. $n=3$, $k=2$.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

Then if $I = (1, 3)$,

$$V_I = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \quad dx_I(\vec{v}_1, \vec{v}_2) = 11$$

and if $I = (2, 2)$

$$V_I = \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix} \quad dx_I(\vec{v}_1, \vec{v}_2) = 0.$$

Proposition. The dimension of $\Lambda^k(\mathbb{R}^n)$ is $\binom{n}{k}$ and a basis for $\Lambda^k(\mathbb{R}^n)$ is given by $\{dx_I\}$ where $I = (i_1, \dots, i_k)$ with $i_1 < i_2 < \dots < i_k$.

Proof. Homework.

Example. $\Lambda^2(\mathbb{R}^3)$ has dimension $\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2} = 3$.

A basis for $\Lambda^2(\mathbb{R}^3)$ is given by

$$dx_{12}, dx_{13}, dx_{23}$$

Definition. The wedge product

$$\wedge : \Lambda^k(\mathbb{R}^n) \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^{k+l}(\mathbb{R}^n)$$

is the unique bilinear map which has

$$dx_I \wedge dx_J = dx_{(I,J)}$$

where if $I = (i_1, \dots, i_k)$ and

$J = (j_1, \dots, j_l)$ then

$$(I, J) = (i_1, \dots, i_k, j_1, \dots, j_l)$$

is the concatenation of I and J .

Examples

$$dx_{12} \wedge dx_3 = dx_{123}$$

$$dx_{17} \wedge dx_{25} = dx_{1725}$$

Important example. Suppose

$$\omega = a_1 dx_1 + a_2 dx_2 \in \Lambda^1(\mathbb{R}^2)$$

$$\eta = b_1 dx_1 + b_2 dx_2 \in \Lambda^1(\mathbb{R}^2).$$

Then we compute (bilinearity)

$$\omega \wedge \eta = a_1 b_1 dx_1 \wedge dx_1$$

$$+ a_1 b_2 dx_1 \wedge dx_2$$

$$+ a_2 b_1 dx_2 \wedge dx_1$$

$$+ a_2 b_2 dx_2 \wedge dx_2$$

$$= a_1 b_1 dx_{11} + a_1 b_2 dx_{12}$$

$$+ a_2 b_1 dx_{21} + a_2 b_2 dx_{22}$$

Now we stop to think.

Exchanging entries in a multi-index exchanges rows

in a determinant. So it changes the sign of the result (alternating). This means

$$dx_{\underbrace{11}} = -dx_{11} = 0$$

$$dx_{\underbrace{21}} = -dx_{12}$$

$$dx_{\underbrace{22}} = -dx_{22} = 0$$

so

$$\omega \wedge \eta = (a_1 b_2 - a_2 b_1) dx_{12}$$

Proposition. The wedge product
 $\wedge: \Lambda^k(\mathbb{R}^n) \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^{k+l}(\mathbb{R}^n)$
is associative and obeys

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

Proof. Associativity for basis elements is formal:

$$(dx_I \wedge dx_J) \wedge dx_K = dx_{(I, J, K)}$$

but (I, J, K) is the concatenation of the multi-indices I, J, K , so it is the same as $(I, (J, K))$.

Thus

$$(dx_I \wedge dx_J) \wedge dx_K = dx_I \wedge (dx_J \wedge dx_K)$$

Now we've already observed that if I' is related to I by swapping a pair of indices,

$$dx_{I'} = -dx_I.$$

So

$$dx_I \wedge dx_J = dx_{IJ}$$

$$= \pm dx_{JI} = dx_J \wedge dx_I$$

where the sign is determined by how many swaps it takes to rearrange

$$i_1 \cdots i_k j_1 \cdots j_\ell \rightarrow j_1 \cdots j_\ell i_1 \cdots i_k$$

but it takes K swaps to move j_1 to the front

$$i_1 \cdots i_k j_1 \cdots j_\ell \rightarrow j_1 i_1 \cdots i_k j_2 \cdots j_\ell$$

and then K more for j_2 , and eventually $K\ell$ total swaps.

Thus

$$\omega \wedge \eta = (-1)^{K\ell} \eta \wedge \omega. \quad \square$$

This allows us to prove

Lemma. If $I = (i_1, \dots, i_k)$ then

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Note the lack of parentheses!

Let's take stock. We've defined vector spaces $\Lambda^k(\mathbb{R}^n)$. We are now going to consider vector valued functions, whose values lie in these spaces.

Definition. If $U \subset \mathbb{R}^n$ is an open set, and $\omega: U \subset \mathbb{R}^n \rightarrow \Lambda^k(\mathbb{R}^n)$ is a smooth function, we say ω is a differential k-form on U .

The set of k -forms $A^k(U)$ is an (infinite-dimensional) vector space,

which just means that we may add and scalar multiply k -forms using the vector space structure on $\Lambda^k(\mathbb{R}^n)$.

Since our basis for $\Lambda^k(\mathbb{R}^n)$ doesn't have a natural order, it's easier to think of "basis functions" for a form instead of "coordinate functions", and write

$$\omega = \sum_{\substack{I \text{ an} \\ \text{increasing} \\ k\text{-index} \\ \text{in } \mathbb{R}^n}} f_I dx_I$$

where $f_I: U \subset \mathbb{R}^n \rightarrow \mathbb{R}$.

Now we're going to write

$$\omega = \sum_I f_I dx_I \in A^k(U)$$

for an open $U \subset \mathbb{R}^n$.

On the right hand side, we know the $\{dx_I\}$ span $\Lambda^k(\mathbb{R}^n)$ so every vector in this vector space can be written as a linear combination of dx_I 's.

The f_I 's are the coefficients.

Actually, a subset of $\{dx_I\}$ (the ones with increasing indices) suffices. But there's no harm in summing over every I if you want (and it simplifies things later).

We now think about notation.

ω : a vector valued function

$$\begin{array}{ccc} \vec{x} & \mapsto & \underbrace{\sum_{\mathbf{I}} f_{\mathbf{I}}(\vec{x}) dx_{\mathbf{I}}}_{\text{in } \Lambda^k(\mathbb{R}^n)} \quad (*) \\ \uparrow & & \uparrow \\ \text{in } U & & \end{array}$$

$\omega(\vec{x})$: an element of $\Lambda^k(\mathbb{R}^n)$,
an alternating, multilinear
function from $(\mathbb{R}^n)^k \rightarrow \mathbb{R}$

$\omega(\vec{x})(\vec{v}_1, \dots, \vec{v}_k)$: a real number.

(*)

When $k=0$,

$\omega(\vec{x})$ is an element

of $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$.

Examples.

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then every $\omega \in A^1(\mathbb{R}^2)$ can be written in the form:

$$\omega = f_1 dx_1 + f_2 dx_2$$

So if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ we have

$$\begin{aligned} \omega(\vec{x})(\vec{v}) &= f_1(\vec{x}) dx_1(\vec{v}) \\ &\quad + f_2(\vec{x}) dx_2(\vec{v}) \end{aligned}$$

$$= f_1(\vec{x}) \cdot v_1 + f_2(\vec{x}) \cdot v_2$$

$$= \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then every $\omega \in \Lambda^2(\mathbb{R}^3)$ can be written in the form:

$$\begin{aligned} \omega &= f_{12} dx_1 \wedge dx_2 \\ &\quad + f_{13} dx_1 \wedge dx_3 \\ &\quad + f_{23} dx_2 \wedge dx_3 \end{aligned}$$

so if we have vectors \vec{v}_1, \vec{v}_2 the value

$$\begin{aligned} \omega(\vec{x})(\vec{v}_1, \vec{v}_2) &= \\ &f_{12}(\vec{x}) dx_1 \wedge dx_2(\vec{v}_1, \vec{v}_2) \\ &+ f_{13}(\vec{x}) dx_1 \wedge dx_3(\vec{v}_1, \vec{v}_2) \\ &+ f_{23}(\vec{x}) dx_2 \wedge dx_3(\vec{v}_1, \vec{v}_2) \end{aligned}$$

$$= f_{12}(\vec{x}) \cdot \det \begin{bmatrix} (\vec{v}_1)_1 & (\vec{v}_2)_1 \\ (\vec{v}_1)_2 & (\vec{v}_2)_2 \end{bmatrix}$$

$$+ f_{13}(\vec{x}) \cdot \det \begin{bmatrix} (\vec{v}_1)_1 & (\vec{v}_2)_1 \\ (\vec{v}_1)_3 & (\vec{v}_2)_3 \end{bmatrix}$$

$$+ f_{23}(\vec{x}) \cdot \det \begin{bmatrix} (\vec{v}_1)_2 & (\vec{v}_2)_2 \\ (\vec{v}_1)_3 & (\vec{v}_2)_3 \end{bmatrix}$$

which is starting to look a
lot like expansion by minors
 in a determinant!

In fact, this is a determinant:

$$\omega(\vec{x})(\vec{v}_1, \vec{v}_2) =$$

$$\det \begin{bmatrix} f_{23}(\vec{x}) & \uparrow & \uparrow \\ -f_{13}(\vec{x}) & \vec{v}_1 & \vec{v}_2 \\ f_{12}(\vec{x}) & \downarrow & \downarrow \end{bmatrix}$$

Note that it's not traditional to write forms with a "vector" superscript, but we certainly

could : $\omega(\vec{x})$ is a vector

(in a vector space of functions)

(so it's also a function).

We define the wedge product for differential forms by bilinearity of wedge, so

Definition. If $\omega \in A^k(U)$, $\eta \in A^l(U)$ for some open $U \subset \mathbb{R}^n$, and

$$\omega = \sum_I f_I dx_I \quad \text{while}$$

$$\eta = \sum_J g_J dx_J, \quad \text{we define}$$

$$\omega \wedge \eta = \sum_{I, J} f_I g_J dx_I \wedge dx_J$$

$$= \sum_{I, J} f_I g_J dx_{IJ}$$

Proposition. If $U \subset \mathbb{R}^n$ is open,
 $\omega \in A^k(U)$, $\eta \in A^l(U)$, $\varphi \in A^m(U)$,
then

1) If $k=l=m$, we have

$$\omega + \eta = \eta + \omega \in A^k(U)$$

$$(\omega + \eta) + \varphi = \omega + (\eta + \varphi) \in A^k(U)$$

$$2) \quad \omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \in A^{k+l}(U)$$

$$3) \quad (\omega \wedge \eta) \wedge \varphi = \omega \wedge (\eta \wedge \varphi) \in A^{k+l+m}(U)$$

4) If $k=l$, then

$$(\omega + \eta) \wedge \varphi = \omega \wedge \varphi + \eta \wedge \varphi$$

All of this seems awfully simple and clean; so much that you might worry that it doesn't tell you much. But there are some nonobvious conclusions already.

Lemma. If $k > n$, then $\Lambda^k(\mathbb{R}^n)$ contains only one form $\omega = 0$.

Proof. The vector space $\Lambda^k(\mathbb{R}^n)$ has a basis consisting of functions dx_I where $I = (i_1, \dots, i_k)$. Since $k > n$, an

index must be repeated. But then $dx_I(\vec{v}_1, \dots, \vec{v}_k) = 0$ for all inputs. Thus $\Lambda^k(\mathbb{R}^n) = \{0\}$.

Now ω is a smooth map $\omega: U \rightarrow \{0\}$, so ω must have the constant value 0. This uniquely determines ω , so there is only a single element in $\Lambda^k(\mathbb{R}^n)$. \square

Differentiating differential forms.

Consider the 0-form. By our definition, if $U \subset \mathbb{R}^n$ is open

$$A^0(U) = \text{smooth functions} \\ \text{from } U \rightarrow \Lambda^0(\mathbb{R}^n) = \mathbb{R}$$

Now we know how to define the derivative: If $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ then each $Df(\vec{x}): \mathbb{R}^n \rightarrow \mathbb{R}$ is a linear function from $\mathbb{R}^n \rightarrow \mathbb{R}$.

If we instead think of $Df(\vec{x})$ as an element of $\Lambda^1(\mathbb{R}^n)$, we are inspired to say

Definition. The exterior derivative of $f \in A^0(U)$ is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n \in A^1(U)$$

If $\omega = \sum_I f_I dx_I \in A^k(U)$, then

$$d\omega = \sum_I df_I \wedge dx_I \in A^{k+1}(U).$$

We note that

$$d\omega = \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Examples.

$$f: \mathbb{R} \rightarrow \mathbb{R}. \quad df = f'(x) dx$$

$$\omega = y dx + x dy \in A^1(\mathbb{R}^2)$$

$$\begin{aligned} d\omega &= \left(\overset{0}{\cancel{\frac{\partial}{\partial x} y}} dx + \overset{1}{\cancel{\frac{\partial}{\partial y} y}} dy \right) \wedge dx \\ &\quad + \left(\overset{1}{\cancel{\frac{\partial}{\partial x} x}} dx + \overset{0}{\cancel{\frac{\partial}{\partial y} x}} dy \right) \wedge dy \\ &= dy \wedge dx + dx \wedge dy = 0. \end{aligned}$$

$$\omega = x_1 dx_2 + x_3 dx_4 + x_5 dx_6 \in A^1(\mathbb{R}^6)$$

$$d\omega = dx_4 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 \in A^2(\mathbb{R}^6)$$

Proposition. Let $\omega \in A^k(U)$, $\eta \in A^l(U)$ and $f \in A^0(U)$ a smooth function.

1) When $k=l$,

$$d(\omega + \eta) = d\omega + d\eta$$

$$2) \quad d(f\omega) = df \wedge \omega + f d\omega$$

$$3) \quad d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

$$4) \quad d(d\omega) = 0.$$

Proof. We start with 1. Suppose f and g are smooth functions $U \rightarrow \mathbb{R}$.

$$d(f+g) = \sum_i \frac{\partial}{\partial x_i} (f+g) dx_i$$

$$= \sum_i \frac{\partial}{\partial x_i} f dx_i + \frac{\partial}{\partial x_i} g dx_i$$

$$= df + dg$$

Now suppose $\omega = \sum_I f_I dx_I$
and $\eta = \sum_I g_I dx_I \in A^k(U)$. $\textcircled{*}$

Then

$$d(\omega + \eta) = d\left(\sum_I (f_I + g_I) dx_I\right)$$

(definition of d)

$$= \sum_I d(f_I + g_I) \wedge dx_I$$

(we just proved it)

$$= \sum_I (df_I + dg_I) \wedge dx_I$$

(linearity of \wedge)

$$= \sum_I df_I \wedge dx_I + dg_I \wedge dx_I$$

(rearranging sums)

$$= d\omega + d\eta.$$

$\textcircled{*}$ Since these are both k -forms,
we are summing over the same
set of increasing k -indices for \mathbb{R}^n ,
so it makes sense to write \sum_I in
each case.

Now we'll prove property 3.

We have proved that d is linear and know that \wedge is bilinear.

So both sides of the proposed equation

$$d(\omega \wedge \eta) \stackrel{?}{=} d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

are linear in ω and in η .

Therefore, it suffices to show

$$\begin{aligned} & d(f dx_I \wedge g dx_J) \\ &= d(fg dx_I \wedge dx_J) \\ &= d(fg) \wedge dx_I \wedge dx_J \\ &= (g df + f dg) \wedge dx_I \wedge dx_J \end{aligned}$$

$$\begin{aligned}
&= g df \wedge dx_I \wedge dx_J \\
&\quad + f dg \wedge dx_I \wedge dx_J \\
&= (df \wedge dx_I) \wedge (g dx_J) \\
&\quad + (-1)^{k+1} f dx_I \wedge dg \wedge dx_J \\
&= d(f dx_I) \wedge g dx_J \\
&\quad + (-1)^k f dx_I \wedge d(g dx_J).
\end{aligned}$$

Note that property 2 follows by the same argument.

We now consider property 4:

$$d(d(\omega)) =$$

Since d is linear, $d \circ d$ is linear. So it suffices to show this in the case $\omega = f dx_I$.

$$\begin{aligned} d\omega &= df \wedge dx_I \\ &= \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I \end{aligned}$$

so

$$\begin{aligned} d(d\omega) &= \sum_{j=1}^n d\left(\frac{\partial f}{\partial x_j}\right) \wedge dx_j \wedge dx_I \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k} dx_k \wedge dx_j \wedge dx_I \end{aligned}$$

$$\text{Now } dx_k \wedge dx_j = (-1)^{k \cdot j} dx_j \wedge dx_k.$$

So we may regroup this sum as

$$= \sum_{j=1}^n \sum_{k=1}^{j-1} \underbrace{\left(\frac{\partial^2 f}{\partial x_j \partial x_k} - \frac{\partial^2 f}{\partial x_k \partial x_j} \right)}_{=0} dx_j \wedge dx_k \wedge dx_I$$

$= 0$, since the functions are smooth (C^∞) they are C^2 and mixed partials commute.

This completes the proof.