Geodesics and Abstract Geometries

We have given the geodesic equations in one form (natural from the point of view of geodesic curvature).

Theorem. $\alpha(s) = x(u(s), v(s))$ a geodesic $\iff$

$$\frac{d}{ds}(Eu' + Fv') = \frac{1}{2}\left(E_u(u')^2 + 2F_u u'v' + G_u(v')^2\right)$$

$$\frac{d}{ds}(Fu' + Gv') = \frac{1}{2}\left(E_v(u')^2 + 2F_v u'v' + G_v(v')^2\right)$$

and $E(u')^2 + Fu'v' + G(v')^2 = 1$.

Proof. Observe

$$\frac{d}{ds}(Eu' + Fv') = (E_u u' + E_v v')u' + Eu'' + (F_u u' + F_v v')v' + Fv''$$

so we have

$$\frac{d}{ds}(Eu' + Fv') - Eu'' - Fv'' = (E_u u' + E_v v')u' + (F_u u' + F_v v')v'$$

adding this to the first geodesic equation

$$u'' E + v'' F = -\frac{1}{2} E_u (u')^2 - E_v u'v' - (F_v - \frac{1}{2} G_u)(v')^2$$

yields

$$\frac{d}{ds}(Eu' + Fv') = \frac{1}{2} E_u (u')^2 + F_u u'v' + \frac{1}{2} G_u (v')^2.$$

The other equation is similar.

The third enforces that α is arclength parametrized. ▨

Since everything is expressed in terms of E, F, G, do we still need X?

<u>Definition</u>. A <u>metric</u> on $U \subset \mathbb{R}^2$ is a smooth function $I : U \to Mat_{2\times 2}(\mathbb{R})$ so that

$$I(\vec{p}) = \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix}$$

is a <u>symmetric</u> <u>positive</u> <u>definite</u> matrix for $\vec{p} \in U$.

Every parametrization $X: U \to \mathbb{R}^3$ induces a metric, because

$$I_p = (DX(\vec{p}))^T (DX(\vec{p}))$$

is s.p.d. by construction.

Example. The upper half-plane with

$$E = G = 1/v^2, \quad F = 0.$$

now $E_u = G_u$ so the equations become

$$\frac{d}{ds}\left(\frac{u'}{v^2}\right) = 0$$

$$\frac{d}{ds}\left(\frac{v'}{v^2}\right) = \frac{1}{\cancel{2}}\left(-\frac{\cancel{2}}{v^3}(u')^2 - \frac{\cancel{2}}{v^3}(v')^2\right)$$

$$\frac{1}{v^2}(u')^2 + \frac{1}{v^2}(v')^2 = 1$$

Now

$$\frac{d}{ds}\left(\frac{u'}{v^2}\right) = 0$$

so there is a constant C_2 so that

$$u' = C_2 v^2.$$

If $C_2 = 0$, then the equations become

$$u(s) = \text{constant}$$

$$\frac{d}{ds}\left(\frac{v'}{v^2}\right) = -\frac{(v')^2}{v^3}$$

$$\frac{(v')^2}{v^2} = 1$$

The third equation is

$v' = \pm v$, which has $v(s) = ce^{\pm s}$

Now if $v(s) = ce^{-s}$, then

$$v'(s) = -ce^{-s}$$

$$\frac{v'(s)}{v^2(s)} = \frac{-ce^{-s}}{c^2 e^{-2s}} = -\frac{1}{c} e^{s}$$

and

$$\frac{d}{ds} \frac{v'(s)}{v^2(s)} = -\frac{1}{c} e^{s}$$

while

$$-\frac{(v')^2}{v^3} = -\frac{c^2 e^{-2s}}{c^3 e^{-3s}} = -\frac{1}{c} e^{s}$$

On the other hand, if $v(s) = ce^{s}$, $v'(s) = ce^{s}$, and we have

$$\frac{v'(s)}{v^2(s)} = \frac{ce^{s}}{c^2 e^{2s}} = \frac{1}{c} e^{-s}$$

so

$$\frac{d}{ds}\left(\frac{v'}{v^2}\right) = -\frac{1}{c} e^{-s}$$

while

$$-\frac{(v')^2}{v^3} = -\frac{c^2 e^{2s}}{c^3 e^{3s}} = -\frac{1}{c} e^{-s}.$$

We conclude that

$$u(s) = c_0, \quad v(s) = c_1 e^{\pm s}$$

is always a geodesic.

Now suppose that $c_2 \neq 0$ and

$$u' = c_2 v^2.$$

Then

$$\frac{(u')^2}{v^2} + \frac{(v')^2}{v^2} = 1 \Rightarrow c_2^2 v^2 + \frac{(v')^2}{v^2} = 1$$

This second and third equation yield

$$\frac{d}{ds}\left(\frac{v'}{v^2}\right) = \frac{1}{\cancel{2}}\left(-\frac{\cancel{2}}{v^3}(u')^2 - \frac{\cancel{2}}{v^3}(v')^2\right)$$

$$= -\frac{1}{v}\left(\cancel{\frac{(u')^2}{v^2} + \frac{(v')^2}{v^2}}\right)^{1}$$

$$= -\frac{1}{v}.$$

Now $\frac{d}{ds}\left(-\frac{1}{v}\right) = \frac{v'}{v^2}$, so we can see that this equation is

$$\frac{d^2}{ds^2}\left(-\frac{1}{v}\right) = -\frac{1}{v} \iff \frac{d^2}{ds^2}\left(\frac{1}{v}\right) = \frac{1}{v}$$

We conclude that (in general)

$$\frac{1}{v} = c_0 \cosh(s + c_1)$$

so

$$v = c_0 \operatorname{sech}(s + c_1), \text{ where } c_0 > 0.$$

and

$$v'(s) = c_0 \operatorname{sech}(s+c_1)\tanh(s+c_1)$$

So

$$\frac{v'}{v} = \tanh(s+c_1)$$

Returning to $u' = c_2 v^2$ and

$$c_2^2 v^2 + \frac{(v')^2}{v^2} = 1$$

we see

$$c_2^2 c_0^2 \operatorname{sech}^2(s+c_1) + \tanh^2(s+c_1) = 1$$

Since $\operatorname{sech}^2 x + \tanh^2 x = 1$, we get

$c_2^2 c_0^2 = 1$, so we conclude

$$u'(s) = \frac{1}{c_0} \cdot c_0^2 \operatorname{sech}^2(s+c_1)$$

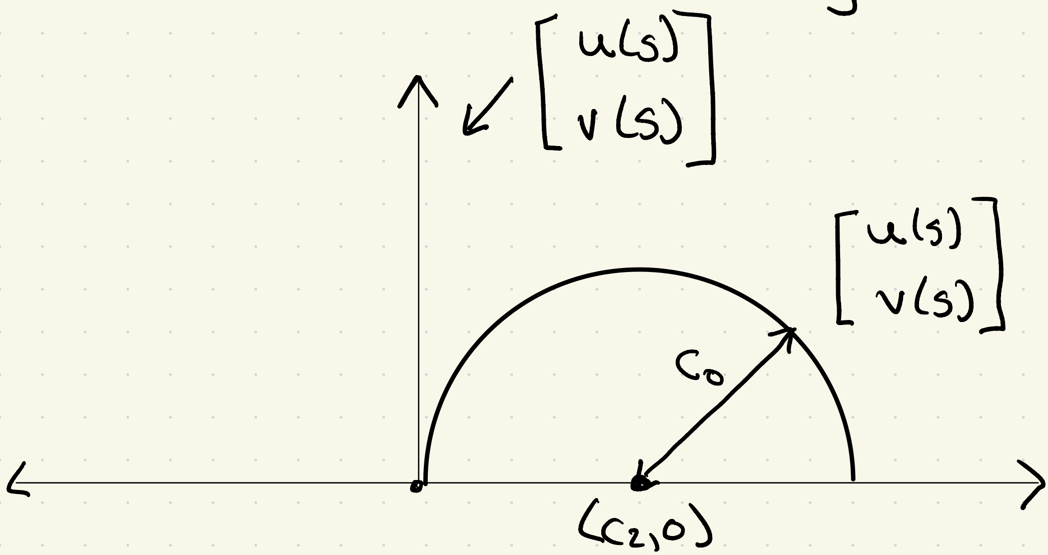
and

$$u(s) = c_0 \tanh(s+c_1) + c_2.$$

Now we just need to understand c_0, c_1, and c_2. Notice that

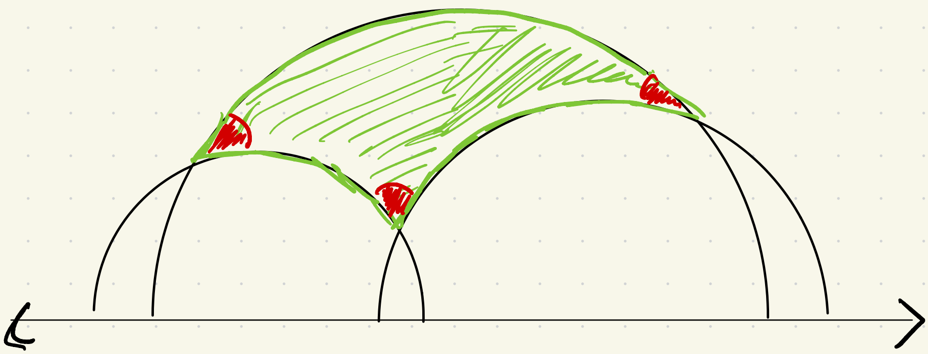
$$\begin{aligned}(u-c_2)^2 + v^2 &= c_0^2 \tanh^2(s+c_1) + c_0^2 \operatorname{sech}^2(s+c_1) \\ &= c_0^2.\end{aligned}$$

Thus the solutions are really nice!

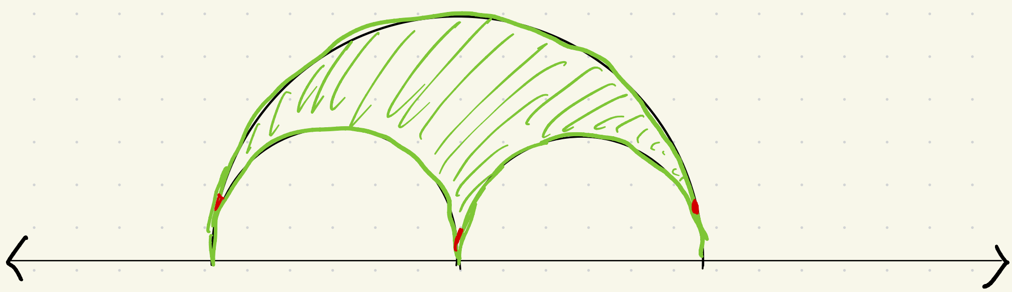


We call this the <u>hyperbolic</u> <u>plane</u>.

Triangles in the hyperbolic plane are interesting:



Angles sum to <u>less</u> than π.



A triangle with angle sum zero!

__Definition.__ If $F: \mathbb{R}^2 \to \mathbb{R}^2$, and I_p is a metric on $\mathbb{R}^2$, we say F is an __isometry__ of I_p if

$$\left(DF(\vec{p})\right)^T I_{F(\vec{p})} \, DF(\vec{p}) = I_{\vec{p}}$$

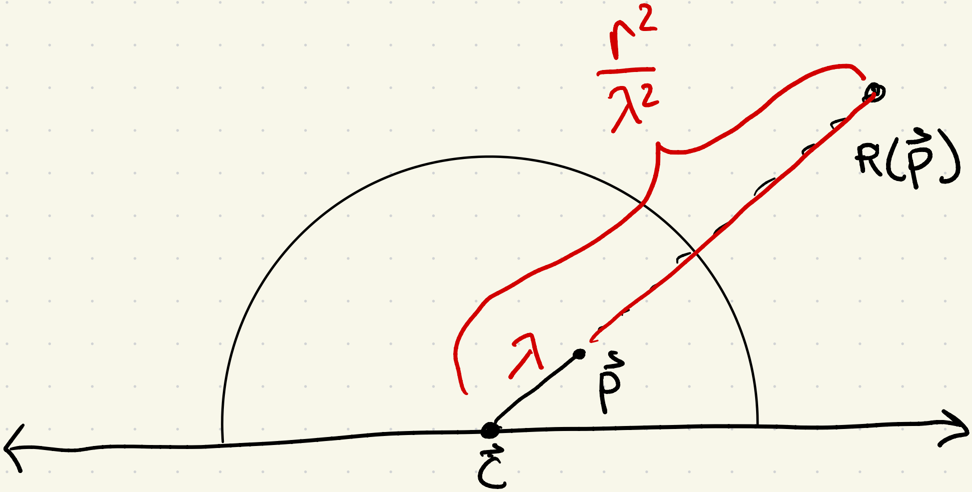
__Example.__ If $F: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $F(\vec{x}) = A\vec{x} + \vec{c}$ where A is an orthogonal matrix and $I_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$DF(\vec{p})^T I \, DF(\vec{p}) = A^T I A = I.$$

So translations and rotations (and reflections) are Euclidean isometries

What about __hyperbolic__ isometries?

Definition. If $\vec{c} \in \mathbb{R}^2$ and $r \in \mathbb{R} > 0$, the map $R(\vec{p}) = \frac{r^2}{\lambda^2}(\vec{p} - \vec{c}) + \vec{c}$, where $\lambda = \|\vec{p} - \vec{c}\|$ is called <u>inversion</u>.



Lemma. $DR(\vec{p}) = \frac{r^2}{\lambda^2} A$, where $A = \left(I - 2\frac{\vec{p}\vec{p}^T}{\lambda^2}\right)$ is reflection over the direction orthogonal to $\vec{p} - \vec{c}$, so A is an orthogonal matrix.

Proposition. R is an isometry of $\mathbb{R}^2$ with the hyperbolic metric $I_p = \begin{bmatrix} 1/v^2 & 0 \\ 0 & 1/v^2 \end{bmatrix}$, where $\vec{p} = \begin{bmatrix} u \\ v \end{bmatrix}$.

Proof. Notice that $I_p = \frac{1}{v^2} I_2$, and the v-coordinate of $R(\vec{p})$ is

$$\langle R(\vec{p}), \vec{e}_2 \rangle = \left\langle \frac{r^2}{\lambda^2}(\vec{p} - \vec{c}) + \vec{c}, \vec{e}_2 \right\rangle$$

$$= \frac{r^2}{\lambda^2} \langle \vec{p}, \vec{e}_2 \rangle = \frac{r^2}{\lambda^2} \cdot v$$

Therefore, $I_{R(\vec{p})} = \frac{\lambda^4}{r^4 v^2}$, so

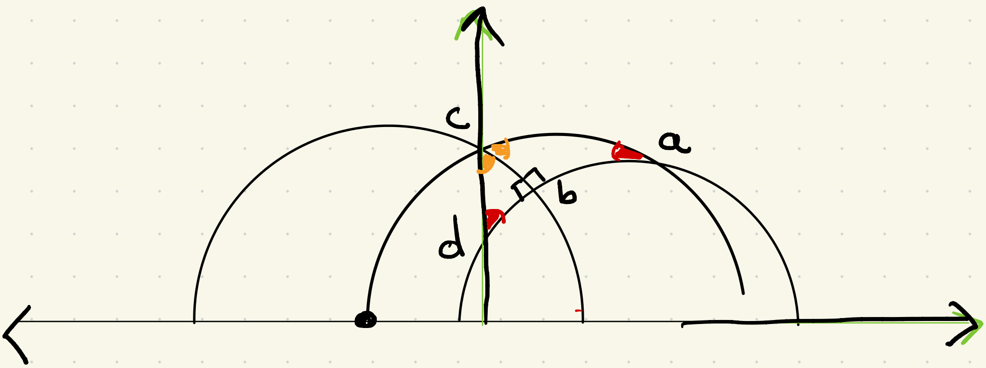
$$DR(\vec{p})^T I_{R(\vec{p})} DR(\vec{p}) = \frac{\cancel{r^2}}{\cancel{\lambda^2}} A^T \frac{\cancel{\lambda^4}}{\cancel{r^4} v^2} I A \frac{\cancel{r^2}}{\cancel{\lambda^2}}$$

$$= \frac{1}{v^2} A^T A = \frac{1}{v^2} I$$

$$= I_{\vec{p}}.$$

$\blacksquare$

These isometries tell us amazing things about the hyperbolic plane.



congruent right triangles

$\Delta abc \cong \Delta dbc$