Geodesics

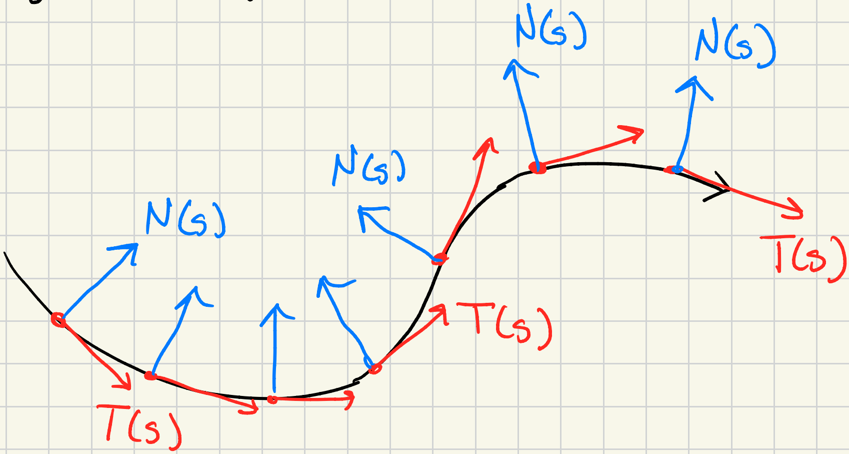
Suppose we have a curve $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^2$, parametrized by arclength.

<u>Definition.</u> If $A: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map "rotate by $+\pi/2$" given by

$$A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix},$$

we let $\vec{x}^\perp = A\vec{x}$.

<u>Definition.</u> The unit tangent vector $T(s) = \vec{\alpha}'(s)$. The unit normal vector $N(s) = T(s)^\perp$.

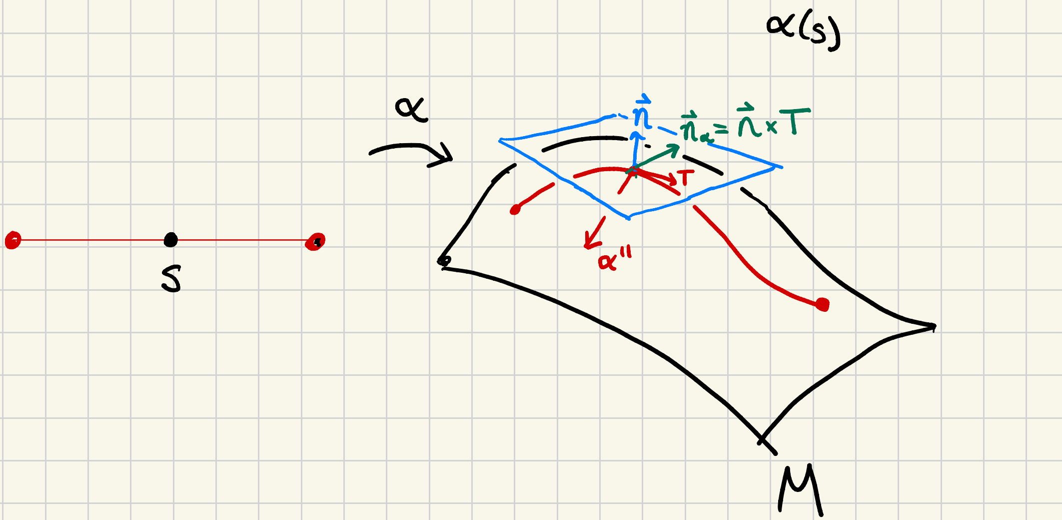


Definition. The Gauss curvature of $\vec{\alpha}(s)$ is defined by $K_{\pm}(s) := \langle \vec{\alpha}''(s), N(s) \rangle$.

positive curvature | zero curvature | negative curvature

The radius of curvature is $1/K_{\pm}(s)$.

Suppose we have a curve on $M \subset \mathbb{R}^3$



We want to do exactly the same construction on this surface.

Definition. If $\vec{n}(\alpha(s))$ is the <u>surface normal</u> at $\alpha(s)$ and $T(s)$ the unit tangent vector $\alpha'(s)$, the <u>intrinsic normal</u> is the unit vector

$$\vec{n}_\alpha(s) = \vec{n}(\alpha(s)) \times T(s)$$

and the <u>geodesic curvature</u> is

$$K_g(s) = \langle \vec{\alpha}''(s), \vec{n}_\alpha(s) \rangle_{\mathbb{R}^3}$$

The <u>normal curvature</u> K_n is given by

$$K_n(s) = \langle \vec{\alpha}''(s), \vec{n}(\alpha(s)) \rangle_{\mathbb{R}^3}$$

Example. If M is the plane, $K_g = K_\pm$ and $K_n = 0$ for any $\alpha(s)$.

Definition. We say $\alpha(s)$ is a <u>geodesic</u> if $\kappa_g(s) \equiv 0$.

Lemma. $\kappa^2 = \kappa_g^2 + \kappa_n^2$, and $\kappa_g \equiv 0 \iff$ $\vec{\alpha}''$ is a scalar multiple of $\vec{n}(\vec{\alpha}(s))$.

Proof. $\vec{n}(\alpha(s))$, $T(s)$ and $\eta_\alpha(s)$ are an orthonormal basis for $\mathbb{R}^3$. Thus

$$\vec{\alpha}'' = \underbrace{\langle \vec{\alpha}'', \vec{n}(\alpha)\rangle}_{\kappa_n} \vec{\eta}(\alpha) + \underbrace{\cancel{\langle \vec{\alpha}'', T\rangle}}_{\substack{0,\text{ since } \alpha'' = \kappa N \\ \text{and } \langle N, T\rangle = 0}} T + \underbrace{\langle \vec{\alpha}'', \vec{\eta}_\alpha\rangle}_{\kappa_g} \vec{\eta}_\alpha.$$

Example. If M is the plane, then $\kappa_n = 0$, and $\kappa_g = \kappa_\pm$. Thus α is a geodesic $\iff$ $\alpha'' = 0$, or α is a straight line.

Example. M = sphere of radius r.

In this case $\vec{n}(\vec{\alpha}) = \dfrac{\vec{\alpha}}{\|\vec{\alpha}\|}$, and so

$$\vec{\alpha}'' = \frac{\chi_n}{\|\vec{\alpha}\|}\vec{\alpha}$$

some scalar

Thus $\vec{\alpha} \times \vec{\alpha}'' \equiv 0$, for all s.

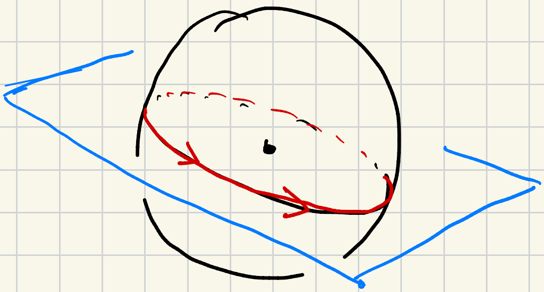
$$0 = \frac{d}{ds}\,\vec{\alpha} \times \vec{\alpha}'' = \vec{\alpha}' \times \vec{\alpha}'' + \vec{\alpha} \times \vec{\alpha}'''$$

Recall from long ago that

$$\tau = \frac{\langle \vec{\alpha}', \vec{\alpha}'' \times \vec{\alpha}''' \rangle}{\|\vec{\alpha}' \times \vec{\alpha}''\|^2} = \frac{\langle \vec{\alpha}''', \vec{\alpha}' \times \vec{\alpha}'' \rangle}{\|\vec{\alpha}' \times \vec{\alpha}''\|^2}$$

$$= -\frac{\cancelto{0}{\langle \vec{\alpha}''', \vec{\alpha} \times \vec{\alpha}''' \rangle}}{\|\vec{\alpha}' \times \vec{\alpha}''\|^2} = 0$$

and that $\gamma \equiv 0 \Rightarrow \alpha$ is a plane curve in the T, N plane. $\Rightarrow$



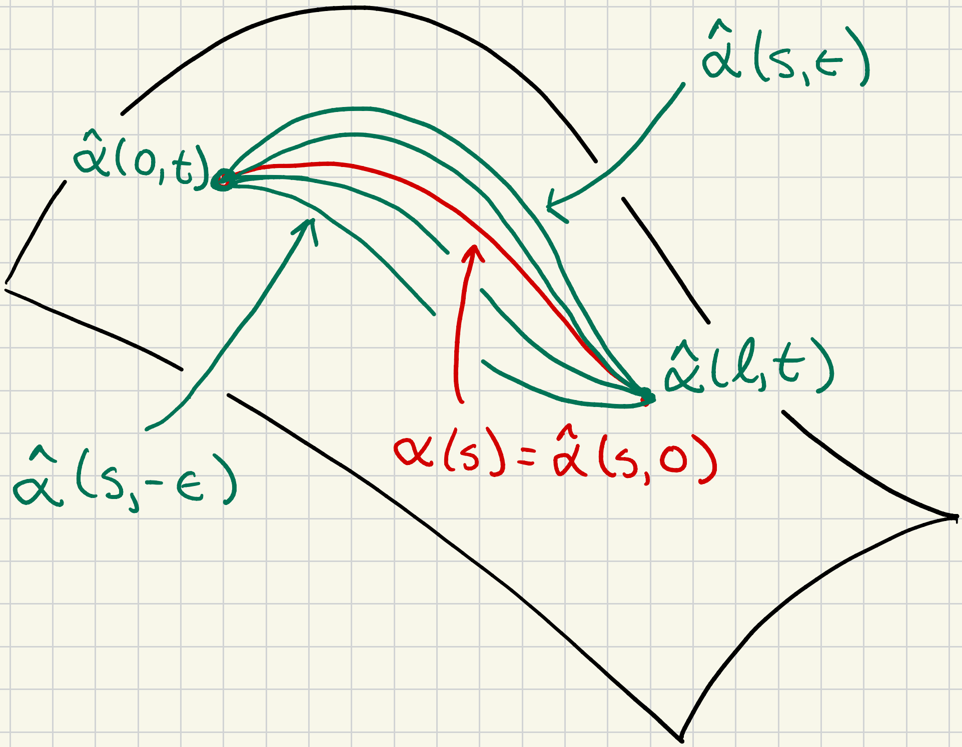
α is a great circle

Definition. If $\alpha : [0,\ell] \to M \subset \mathbb{R}^3$ is arclength parametrized curve, we say α is length-critical if for every C^∞ extension $\hat{\alpha} : [0,\ell] \times [-\epsilon,\epsilon] \to M$

$$\hat{\alpha}(s,0) = \alpha(s), \ \hat{\alpha}(0,t) = \alpha(0), \ \hat{\alpha}(\ell,t) = \alpha(\ell)$$

we have

$$\frac{d}{dt} \mathrm{Length}(\hat{\alpha}(s,t)) \Big|_{t=0} = 0$$



Length-critical curves are stationary positions for rubber bands in the surface.

Theorem. An arclength-parametrized $\alpha : [0,\ell] \to M \subset \mathbb{R}^3$ is length-critical if and only if α is a geodesic.

Proof. We start with a calculation.

$$\frac{d}{dt} \text{Length}(\hat{\alpha}(s,t)) = \frac{d}{dt} \int_0^\ell \left\| \frac{d}{ds} \hat{\alpha}(s,t) \right\| ds$$

$$= \int_0^\ell \frac{d}{dt} \langle \hat{\alpha}_s, \hat{\alpha}_s \rangle^{1/2} ds$$

$$= \int_0^\ell \frac{1}{\cancel{2}} \langle \hat{\alpha}_s, \hat{\alpha}_s \rangle^{-1/2} \left(\cancel{2} \langle \hat{\alpha}_s, \hat{\alpha}_{st} \rangle \right) ds$$

Evaluating at $t=0$, we note

$$\langle \hat{\alpha}_s(s,0), \hat{\alpha}_s(s,0) \rangle = \langle \alpha'(s), \alpha'(s) \rangle$$

$$= \underline{1}$$

Further, $\hat{\alpha}_{st} = \hat{\alpha}_{ts}$. So

$$\frac{d}{dt} \text{Length}(\hat{\alpha}(s,t))\Big|_{t=0}$$

$$= \int_0^{\ell} \left\langle \alpha'(s), \frac{d}{ds}\hat{\alpha}_t(s,0) \right\rangle ds$$

Note that since $\hat{\alpha}(s,t) \in M$ for all t,

$$\hat{\alpha}_t(s,0) \in T_{\hat{\alpha}(s,0)}M.$$

so $\underbrace{\hat{\alpha}_t(s,0)}_{\text{a function of } s.} : [0,\ell] \to T_{\hat{\alpha}(s,0)}M \subset \mathbb{R}^3.$

Now

$$\frac{d}{ds}\left\langle \alpha'(s), \hat{\alpha}_t(s,0) \right\rangle = \left\langle \alpha''(s), \hat{\alpha}_t(s,0) \right\rangle$$

$$+ \left\langle \alpha'(s), \frac{d}{ds}\hat{\alpha}_t(s,0) \right\rangle$$

So we can integrate by parts,

$$= \int_0^\ell \underbrace{\frac{d}{ds}\langle \alpha'(s), \hat{\alpha}_t(s,0)\rangle}_{\langle \alpha'(\ell), \hat{\alpha}_t(\ell,0)\rangle - \langle \alpha'(0), \hat{\alpha}_t(0,0)\rangle} - \langle \alpha''(s), \hat{\alpha}_t(s,0)\rangle \, ds$$

$$\langle \alpha'(\ell), \hat{\alpha}_t(\ell,0)\rangle - \langle \alpha'(0), \hat{\alpha}_t(0,0)\rangle$$

but we assumed $\hat{\alpha}(\ell,0) = \alpha(\ell)$, and $\hat{\alpha}(0,0) = \alpha(0)$, so $\hat{\alpha}_t(\ell,0) = 0$ and $\hat{\alpha}_t(0,0) = 0$.

$$= -\int_0^\ell \langle \alpha''(s), \hat{\alpha}_t(s,0)\rangle \, ds.$$

($\Leftarrow$) Suppose α is a geodesic.

Then $K_g \equiv 0$, so α'' is a scalar multiple of $\eta(\alpha(s))$.

But then $\langle \alpha'', \hat{\alpha}_t(s,0) \rangle \equiv 0$,

because $\hat{\alpha}_t(s,0) \in T_{\alpha(s)}M$, so

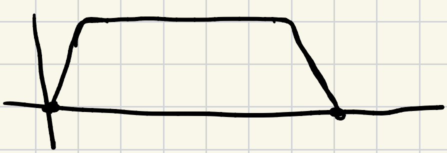
$$\frac{d}{dt} \text{Length}(\hat{\alpha}(s,t)) \Big|_{t=0} = 0.$$

Because $\hat{\alpha}$ was arbitrary, this proves α is length-critical.

($\Rightarrow$) Suppose α is length-critical. For any $\epsilon > 0$, may construct $\hat{\alpha}(t,s)$

$$\hat{\alpha}_t(s,0) = f(s) K_g(s)\, \eta_\alpha(s)$$

where $f \in [0,1]$ on $[0,\ell]$, $f(0) = f(\ell) = 0$ and $f = 1$ on $(\epsilon, \ell - \epsilon)$, and f is smooth.



Then

$$0 = \frac{d}{dt} \text{Length}(\hat{\alpha}(s,t)) \Big|_{t=0} = -\int_0^\ell \langle \alpha'', \hat{\alpha}_t(s,0) \rangle \, ds$$

$$= -\int_0^\ell \langle \alpha'', f_\epsilon(s) K_g(s) \eta_\alpha(s) \rangle \, ds$$

$$= -\int_0^\ell f_\epsilon(s) K_g^2(s) \, ds.$$

It now follows from dominated conv. theorem (or Arzela-Ascoli) that

$$0 = \lim_{\epsilon \to 0} \int_0^\ell f_\epsilon(s) K_g^2(s) \, ds$$

$$= \int_0^\ell K_g^2(s) \, ds \quad \Rightarrow \quad K_g(s) \equiv 0.$$

$\blacksquare$

We are now going to work out a general differential equation for geodesics.

Lemma. If $E = \langle x_u, x_u \rangle$, $F = \langle x_u, x_v \rangle$ and $G = \langle x_v, x_v \rangle$, then

$$\langle x_{uu}, x_u \rangle = \frac{1}{2} E_u$$
$$\langle x_{uu}, x_v \rangle = F_u - \frac{1}{2} E_v$$

$$\langle x_{uv}, x_u \rangle = \frac{1}{2} E_v$$
$$\langle x_{uv}, x_v \rangle = \frac{1}{2} G_u$$

$$\langle x_{vv}, x_u \rangle = F_v - \frac{1}{2} G_u$$
$$\langle x_{vv}, x_v \rangle = \frac{1}{2} G_v$$

Proposition. If $\alpha(s) = x(u(s), v(s))$ is parametrized by arclength, then α is a geodesic $\Leftrightarrow$

$$\frac{d^2}{ds^2}\begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = f\left(\begin{bmatrix} u(s) \\ v(s) \end{bmatrix}, \begin{bmatrix} u'(s) \\ v'(s) \end{bmatrix} \right)$$

where f is smooth, determined by E, F, G.

Proof. We start with

$$\alpha'(s) = x_u u' + x_v v'$$

so

$$\alpha''(s) = (x_{uu} u' + x_{uv} v') u' + x_u u''$$

$$+ (x_{vu} v' + x_{vv} v') v' + x_v v''$$

$$= u'' x_u + v'' x_v +$$

$$(u')^2 x_{uu} + 2u'v' x_{uv} + (v')^2 x_{vv}$$

Now $\alpha(s)$ is a geodesic $\Leftrightarrow$ α'' is a scalar multiple of $n(\alpha) \Leftrightarrow$ $\langle \alpha'', x_u \rangle = \langle \alpha'', x_v \rangle = 0$.

This gives us the equations

$$u'' E + v'' F + \frac{1}{2} E_u (u')^2 + E_v u'v' + (F_v - \tfrac{1}{2} G_u)(v')^2 = 0$$

$$u'' F + v'' G + (F_u - \tfrac{1}{2} E_v)(u')^2 + G_u u'v' + \tfrac{1}{2} G_v (v')^2 = 0$$

Solving for u'' and v'', we get

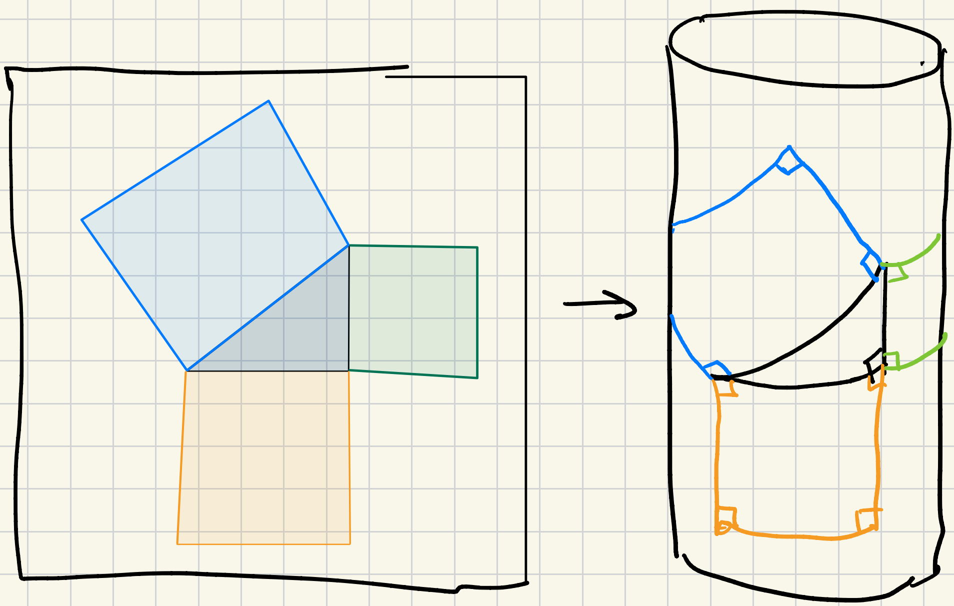
$$\begin{bmatrix} u'' \\ v'' \end{bmatrix} = \underbrace{\frac{-1}{2(EG-F^2)} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} E_u & 2E_v & 2F_v - G_u \\ 2F_u - E_v & 2G_u & G_v \end{bmatrix}}_{\text{a smooth matrix function of } (u,v) \text{, determined by } I_p} \begin{bmatrix} (u')^2 \\ u'v' \\ (v')^2 \end{bmatrix}$$

Corollary. If $F=0$, the equation is

$$\begin{bmatrix} u'' \\ v'' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -E_u/E & -2E_v/E & G_u/E \\ E_v/G & -2G_u/G & -G_v/G \end{bmatrix} \begin{bmatrix} (u')^2 \\ u'v' \\ (v')^2 \end{bmatrix}$$

This tells us something important:

Corollary. If $f: M_0 \to M_1$ is an isometry ($I_p M_0 = I_{f(p)} M_1$) and $\alpha_0(s) = X_0(u(s), v(s))$ is a geodesic on M_0, then $\alpha_1(s) = f(X_0(u(s), v(s)))$ is a geodesic on M_1.



<u>Corollary</u>. If $M \subset \mathbb{R}^3$ is a smooth surface, $\vec{p} \in M$, and $\vec{v} \in T_pM$, there is a unique geodesic $\alpha : [0, \epsilon) \to M$ with $\alpha(0) = \vec{p}$, $\alpha'(0) = \vec{v}$.

Now suppose that M is a surface of revolution

$$x(u,v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where $f'(u)^2 + g'(u)^2 = 1$. Recall that a <u>parallel</u> is a curve $u = c$.

<u>Theorem</u>. If M is a surface of revolution and $\alpha(s) = x(u(s), v(s))$ is unit speed,

$$\alpha(s) \text{ a geodesic} \Rightarrow \langle \alpha', x_v \rangle = \text{constant}$$

$$\left.\begin{array}{l} \langle \alpha', x_v \rangle = \text{constant} \\ \langle \alpha', x_u \rangle \neq 0 \end{array}\right. \Rightarrow \alpha(s) \text{ a geodesic.}$$

Now as we saw before,

$$X_u = (f'(u)\cos v, f'(u)\sin v, g'(u))$$
$$X_v = (-f(u)\sin v, f(u)\cos v, 0)$$

so

$$E = \langle X_u, X_u \rangle = 1 \qquad E_u = E_v = 0$$
$$F = \langle X_u, X_v \rangle = 0 \qquad F_u = F_v = 0$$
$$G = \langle X_v, X_v \rangle = f^2 \qquad G_u = 2ff' \quad G_v = 0.$$

and the geodesic equations are

$$\begin{bmatrix} u'' \\ v'' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -E_u/E & -2E_v/E & G_u/E \\ E_v/G & -2G_u/G & -G_v/G \end{bmatrix} \begin{bmatrix} (u')^2 \\ u'v' \\ (v')^2 \end{bmatrix}$$

$$= \begin{bmatrix} ff'(v')^2 \\ -2\frac{f'}{f}u'v' \end{bmatrix}$$

The second can be rewritten as

$$\frac{v''}{v'} = -\frac{2f'(u)u'}{f(u)}$$

Integrating wrt s on both sides

$$\ln v'(s) = -2\ln f(u(s)) + C$$

so

$$v'(s) = \frac{c}{f(u(s))^2}$$

and

$$f(u(s))^2 v'(s) = c$$

or (since $G = \langle x_v, x_v \rangle = f^2$),

$$\langle x_v, x_v v' \rangle = c$$

But $\alpha' = x_u u' + x_v v'$ and $\langle x_u, x_v \rangle = 0$, so this implies

$$\langle x_v, \alpha' \rangle = c$$

which proves part 1. "If α is a geodesic, then $\langle x_v, \alpha' \rangle = c$."

Now suppose $\langle x_v, \alpha' \rangle = c$ and $\langle x_u, \alpha' \rangle \neq 0$.

Since $\langle x_v, \alpha' \rangle = \langle x_v, x_v v' + x_v u' \rangle = f^2 v'$,

$$f^2 v' = c \Rightarrow 2ff'u'v' + f^2 v'' = 0$$

$$\Rightarrow 2f'u'v' + fv'' = 0$$

Solving for v'' yields

$$v'' = -2\frac{f'}{f}u'v'$$

which is the first geodesic equation!

We know

$$1 = \langle \alpha', \alpha' \rangle = \langle X_u u' + X_v v', X_u u' + X_v v' \rangle$$

$$= E(u')^2 + 2F(u'v') + G(v')^2$$

$$= (u')^2 + (f(u)\, v')^2.$$

Differentiating, we get

$$0 = 2u'u'' + 2f(u)\, v'(f'(u)\, u'v' + f(u) v'')$$

so

$$0 = u'u'' + fv'(f'u'v' + fv'')$$

from the previous argument,

$$f'u'v' + fv'' = -f'u'v'$$

so

$$0 = u'u'' - ff'u'(v')^2$$

$$= u'(u'' - ff'(v')^2)$$

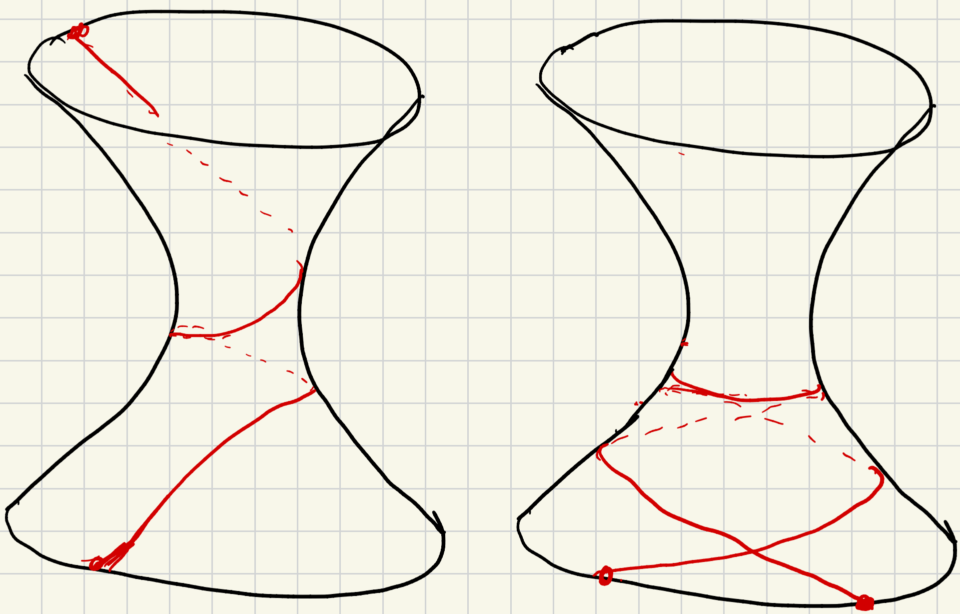
Now by hypothesis,

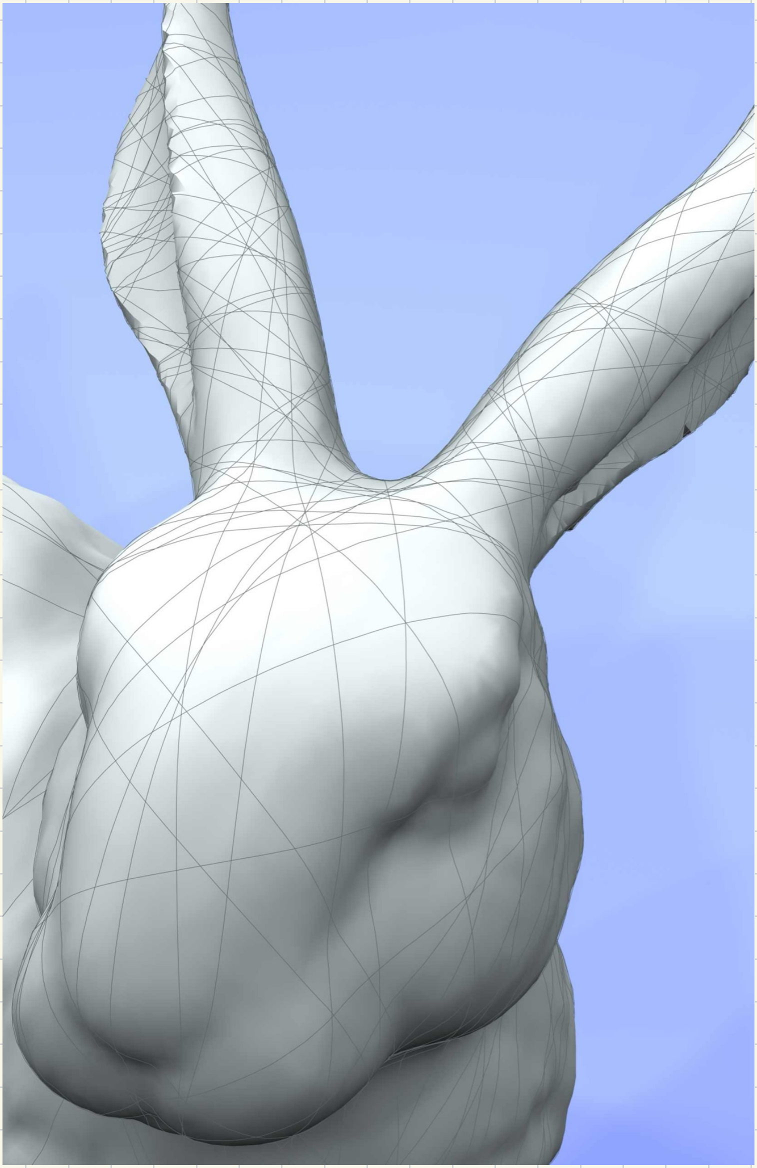
$$\langle x_u, \alpha' \rangle = \langle x_u, x_u u' + x_v v' \rangle$$
$$= \langle x_u, x_u \rangle u'$$
$$= u' \neq 0$$

Therefore,

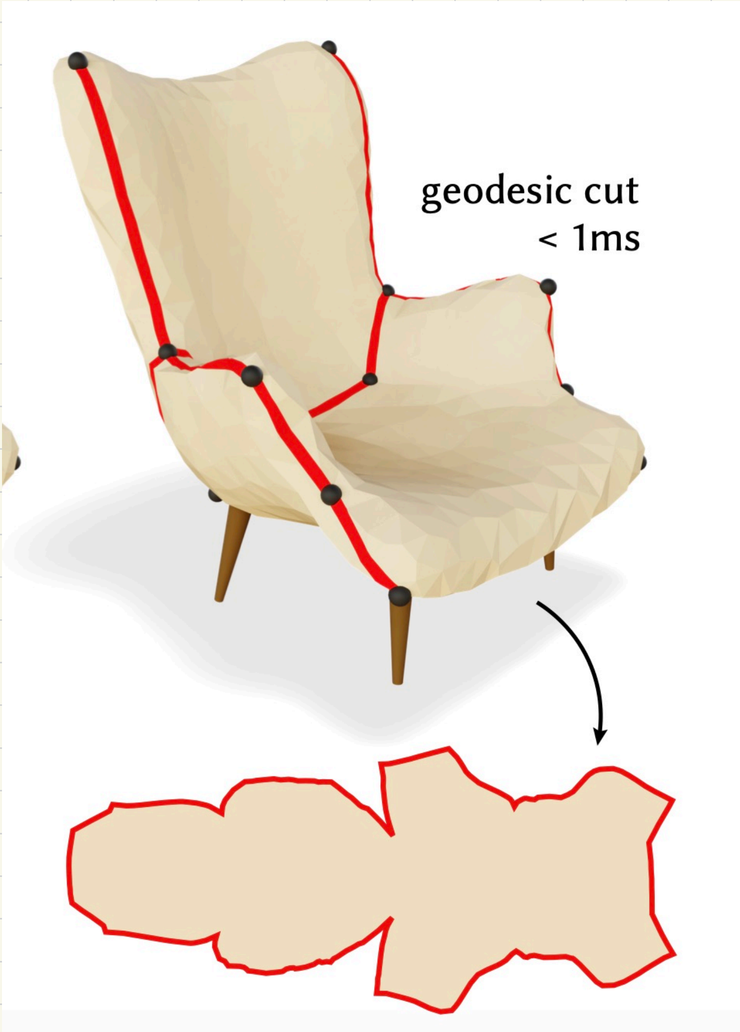
$$u'' = f f' (v')^2,$$

which is the second geodesic equation.





Example of geodesics computed on a triangle mesh.



Geodesics are also used in digital fabrication for pattern cutting.

(Sharp and Crane, 2020)