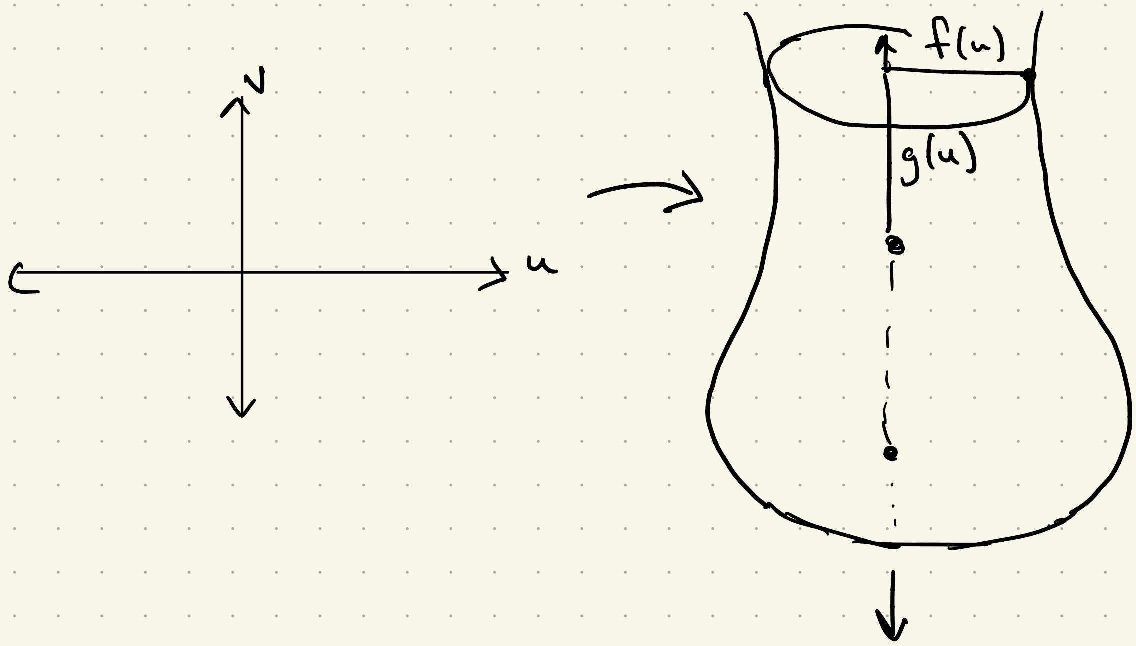
The Space Forms

We now return to a surface we've considered before:



$$\vec{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

$$\vec{X}_u = (f'\cos v, f'\sin v, g')$$

$$\vec{X}_v = (-f\sin v, f\cos v, 0)$$

$$E = (f')^2 + (g')^2, \quad F = 0, \quad G = f^2$$

$$\vec{n} = \frac{(-g'f\cos v, -g'f\sin v, f'f)}{\sqrt{(g')^2 f^2 + (f')^2 f}}$$

$$= \frac{(-g'\cos v, -g'\sin v, f')}{\sqrt{(f')^2 + (g')^2}} \cdot \operatorname{sign}(f)$$

Now

$$\vec{X}_{uu} = (f'' \cos v, f'' \sin v, g'')$$

$$\vec{X}_{uv} = (-f' \sin v, f' \cos v, 0)$$

$$\vec{X}_{vv} = (-f \cos v, -f \sin v, 0)$$

so we have

$$\ell = \langle \vec{n}, \vec{X}_{uu} \rangle = \frac{-g'f''\cos^2 v - g'f''\sin^2 v + f'g''}{\sqrt{(f')^2 + (g')^2}} \cdot \mathrm{sign}(f)$$

$$= \frac{f'g'' - f''g'}{\sqrt{(f')^2 + (g')^2}} \cdot \mathrm{sign}(f)$$

$$m = \langle \vec{n}, \vec{X}_{uv} \rangle = \frac{\cancel{g'f''\cos v \sin v - g'f''\sin v \cos v}}{\sqrt{(f')^2 + (g')^2}}$$

$$= 0$$

$$n = \langle \vec{n}, \vec{x}_{vv} \rangle = \frac{+g'f\cos^2 v + g'f\sin^2 v}{\sqrt{(f')^2 + (g')^2}} \quad \text{sign}(f)$$

$$= \frac{g'f}{\sqrt{(f')^2 + (g')^2}} \quad \text{sign}(f)$$

We can now compute

$$S_p = (\mathrm{I}_p)^{-1} \mathrm{II}_p$$

so

$$K = \det(S_p) = \frac{\ell n - m^2}{EG - F^2}$$

$$= \frac{1}{((f')^2 + (g')^2) f^{\cancel{2}}} \cdot \frac{(f'g'' - f''g')g'\cancel{f}}{(f')^2 + (g')^2}$$

$$= \frac{f'g'g'' - f''(g')^2}{f((f')^2 + (g')^2)^2}$$

If $(f')^2+(g')^2=1$, we also have

$2f'f''+2g'g''=0$ or $f'f''=-g'g''$.

In this case,

$$K=\frac{-(f')^2f''-(g')^2f''}{f\cancel{((f')^2+(g')^2)^2}}=-\frac{f''}{f}$$

Now

$$H=\frac{1}{2}\operatorname{tr}S_p=\frac{En-2Fm+Gl}{2(EG-F^2)}$$

$$=\frac{((f')^2+(g')^2)g'\cancel{f}+f^{\cancel{2}}(f'g''-f''g')}{2((f')^2+(g')^2)f^{\cancel{2}}((f')^2+(g')^2)^{1/2}}\,\mathrm{sign}(f)$$

$$=\frac{((f')^2+(g')^2)g'+f(f'g''-f''g')}{2f((f')^2+(g')^2)^{3/2}}\,\mathrm{sign}(f)$$

Again, <u>if</u> $(f')^2 + (g')^2 = 1$, we can simplify, obtaining

$$H = \frac{g' + f(f'g'' - f''g')}{2f} \, \mathrm{sign}(f)$$

However, it's more useful to simplify using the assumption $g(u) = u$, in which case

$$H = \frac{(1 + (f')^2) - f f''}{2f(1 + (f')^2)^{3/2}} \, \mathrm{sign}(f)$$

Example. Consider the sphere of radius r, with $f(u) = r\cos\left(\frac{u}{r}\right)$ and $g(u) = r\sin\left(\frac{u}{r}\right)$.

Since $f'(u) = -\sin\left(\frac{u}{r}\right)$ and $g'(u) = \cos\left(\frac{u}{r}\right)$, we have

$$f'(u)^2 + g'(u)^2 = 1$$

and may compute using the simplified formulas

$$K = -\frac{f''}{f} = -\frac{-\frac{1}{r}\cancel{\cos\left(\frac{u}{r}\right)}}{r\cancel{\cos\left(\frac{u}{r}\right)}} = \frac{1}{r^2}$$

$$H = \frac{g' + f(f'g'' - f''g')}{2f} \text{sign}(f)$$

Now

$$f'g'' - f''g' = -\sin\left(\frac{u}{r}\right)\left(-\frac{1}{r}\sin\frac{u}{r}\right) - \left(-\frac{1}{r}\cos\frac{u}{r}\right)\left(\cos\frac{u}{r}\right)$$

$$= \frac{1}{r}$$

So

$$H = \frac{\cos\left(\frac{u}{r}\right) + \cancel{r}\cos\left(\frac{u}{r}\right)\cdot\frac{1}{\cancel{r}}}{2r\cos\left(\frac{u}{r}\right)} = \frac{1}{r}$$

Example. The plane. Suppose $f(u) = u$ while $g(u) = 0$. Again,

$$f'(u)^2 + g'(u)^2 = 1^2 + 0^2 = 1$$

so we use the simplified formulas to compute

$$K = \frac{-f''(u)}{f(u)} = \frac{0}{u} = 0$$

$$H = \frac{g' + f(f'g'' - f''g')}{2f} \text{ sign}(f)$$

$$= \frac{0 + u(0 \cdot 0 - 0 \cdot 0)}{2u} = 0$$

Example. The Pseudosphere.

A long time ago, we parametrized the tractrix by

$$\vec{\alpha}(t) = (t - \tanh t, \operatorname{sech} t)$$

Recalling that

$$\cosh^2 t - \sinh^2 t = 1$$

so

$$1 - \tanh^2 t = \operatorname{sech}^2 t$$

while

$$\frac{d}{du} \tanh t = \operatorname{sech}^2 t$$

$$\frac{d}{du} \operatorname{sech} t = -\operatorname{sech} t \tanh t$$

We can compute

$$(f')^2 + (g')^2 = (-\operatorname{sech} t \tanh t)^2 + (1 - \operatorname{sech}^2 t)^2$$

$$= \operatorname{sech}^2 t \tanh^2 t + \tanh^4 t$$

$$= \tanh^2 t (\tanh^2 t + \operatorname{sech}^2 t)$$

$$= \tanh^2 t$$

So we know that the arclength

$$s(t) = \int_0^t \tanh x \, dx$$

$$= \ln(\cosh t)$$

and we can reparametrize by arclength by writing

$$t(s) = \operatorname{arccosh}(e^s)$$

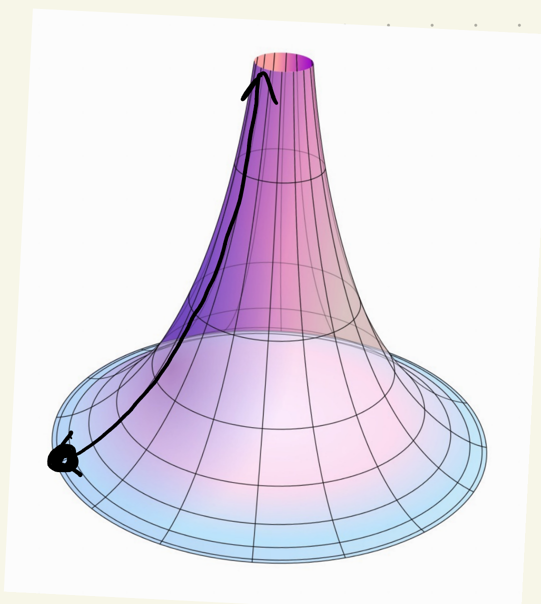
and substituting

to get

$$\vec{\alpha}(s) = \left(\sim\sim ,\ \cosh(\operatorname{arccosh} e^{s}) \right)$$

$$\left(\sim\sim ,\ e^{s} \right)$$

where $\sim$ is some messy function we dislike. Now we can construct a surface by revolving the tractrix

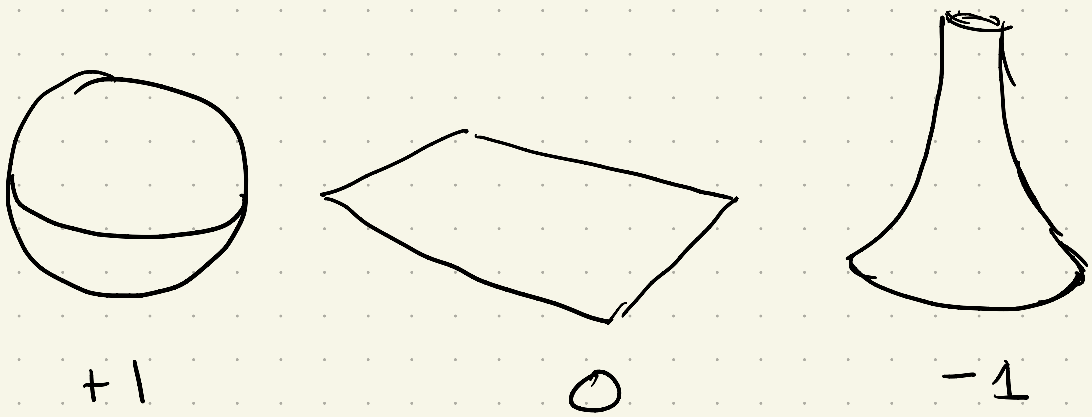


and letting $f(u) = e^{u}$, $g(u) = \sim\sim$. We built f and g so that $(f')^2 + (g')^2 = 1$, so

$$K = -\frac{f''}{f} = -1$$

The mean curvature involves g, so it's messy, and we won't compute it.

We have seen three surfaces of constant curvature:



These are called the <u>space forms</u> as they model three different geometries for a 2-dimensional space.