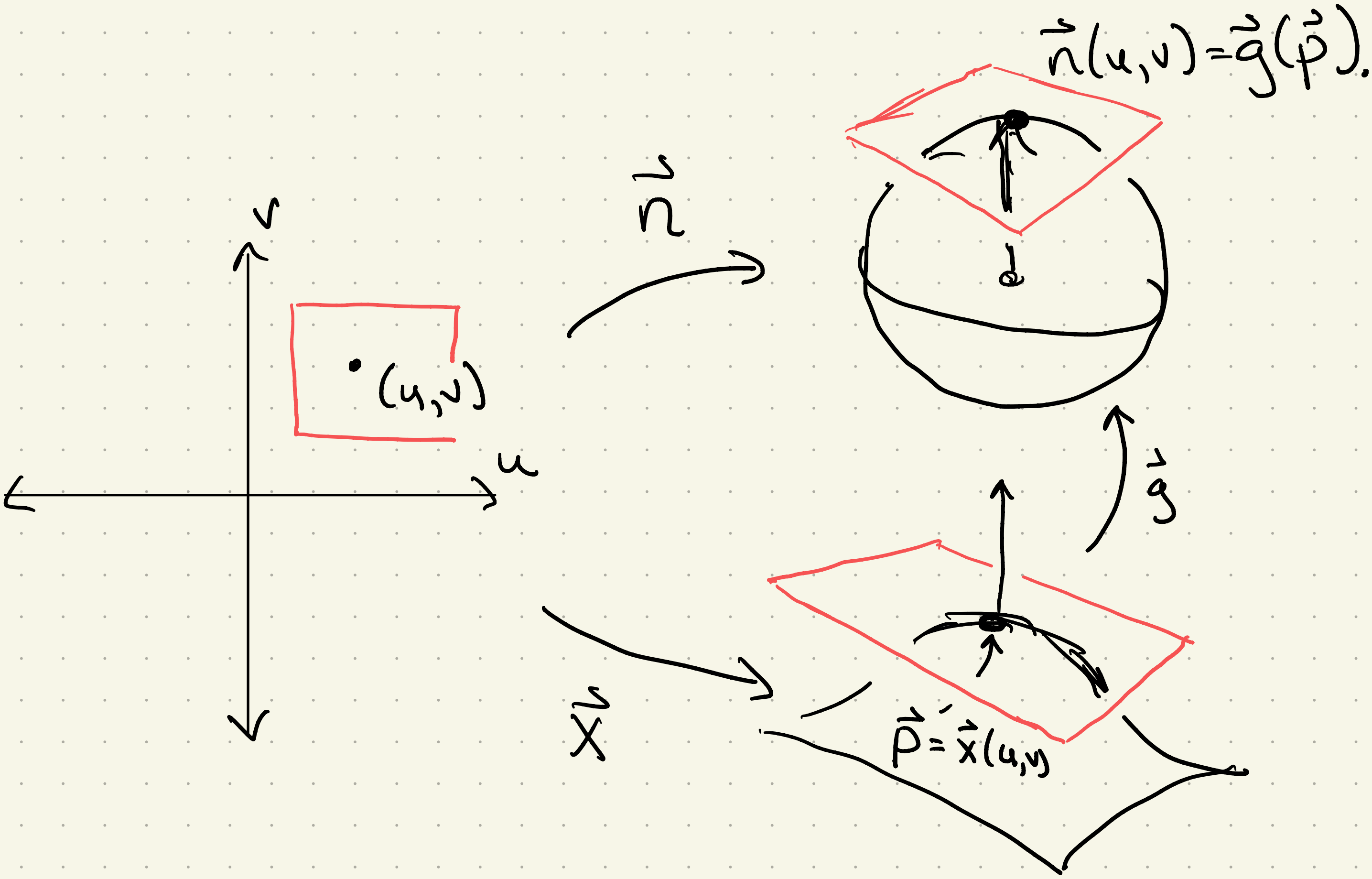
The Gauss Map and Second Fundamental Form

Definition. The normal map $\vec{n}: \mathbb{R}^2 \to \mathbb{R}^3$ is given by $\vec{n} = \dfrac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$.

Definition. The Gauss map $\vec{g}: M \to S^2$ maps each $\vec{p}$ in M to the normal vector to the tangent plane $T_{\vec{p}}M$.



The Gauss map $\vec{g}$, normal map $\vec{n}$, and parametrization $\vec{x}$ are related by

$$\vec{n}(u,v) = \vec{g}(\vec{x}(u,v)).$$

<u>Lemma</u>. The tangent plane $T_{\vec{g}(\vec{p})}S^2 = T_p M$.

Proof. On the sphere, the normal vector is equal to the position vector. So $T_{\vec{g}(\vec{p})}S^2$ is the plane normal to $\vec{g}(\vec{p})$. By definition, this is $T_{\vec{p}}M$. ▨

<u>Definition</u>. The <u>shape operator</u> $S_{\vec{p}} : T_{\vec{p}}M \to T_{\vec{p}}M$ is defined by

$$S_{\vec{p}} = -D\vec{g}(\vec{p})$$

Proposition. The shape operator is the (unique) linear map so that

$$S_{\vec{p}}(\vec{x}_u) = -\vec{n}_u, \quad S_{\vec{p}}(\vec{x}_v) = -\vec{n}_v.$$

Proof. Differentiating $\vec{n}(u,v) = \vec{g}(\vec{x}(u,v))$,

$$\vec{n}_u = D\vec{g}(\vec{x}(u,v))\vec{x}_u = -S_{\vec{p}}(\vec{x}_u)$$
$$\vec{n}_v = D\vec{g}(\vec{x}(u,v))\vec{x}_v = -S_{\vec{p}}(\vec{x}_v)$$

by the chain rule. ▨

Therefore, if $\vec{n}_u = a\vec{x}_u + c\vec{x}_v$, $\vec{n}_v = b\vec{x}_u + d\vec{x}_v$

$$S_{\vec{p}} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

but what are $a, b, c,$ and d?

Lemma. $\langle \vec{x}_u, \vec{n}_v \rangle = \langle \vec{x}_v, \vec{n}_u \rangle$.

Proof. Since $\langle \vec{x}_u, \vec{n} \rangle \equiv 0$, we have

$$0 = \frac{\partial}{\partial v} \langle \vec{x}_u, \vec{n} \rangle$$

$$= \langle \vec{x}_{uv}, \vec{n} \rangle + \langle \vec{x}_u, \vec{n}_v \rangle$$

Since $\langle \vec{x}_v, \vec{n} \rangle \equiv 0$, we have

$$0 = \frac{\partial}{\partial u} \langle \vec{x}_v, \vec{n} \rangle$$

$$= \langle \vec{x}_{vu}, \vec{n} \rangle + \langle \vec{x}_v, \vec{n}_u \rangle$$

But $\vec{x}_{vu} = \vec{x}_{uv}$, so we can subtract

$$0 = \langle \vec{x}_u, \vec{n}_v \rangle - \langle \vec{x}_v, \vec{n}_u \rangle. \qquad \blacksquare$$

Definition. We define

$$\mathrm{II}_{\vec{p}} = \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = -\begin{bmatrix} \langle \vec{n}_u, \vec{X}_u \rangle & \langle \vec{n}_u, \vec{X}_v \rangle \\ \langle \vec{n}_v, \vec{X}_u \rangle & \langle \vec{n}_v, \vec{X}_v \rangle \end{bmatrix}$$

Proposition. $S_{\vec{p}} = (\mathrm{I}_{\vec{p}})^{-1}(\mathrm{II}_{\vec{p}})$.

Proof. We know $S_{\vec{p}} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so

$$-\mathrm{I}_{\vec{p}} S_{\vec{p}} = \begin{bmatrix} \leftarrow X_u \rightarrow \\ \leftarrow X_v \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ X_u & X_v \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} \leftarrow \vec{X}_u \rightarrow \\ \leftarrow \vec{X}_v \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ a\vec{X}_u + c\vec{X}_v & bX_u + dX_v \\ \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \leftarrow \vec{X}_u \rightarrow \\ \leftarrow \vec{X}_v \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ n_u & n_v \\ \downarrow & \downarrow \end{bmatrix} = -\mathrm{II}_p.$$

$\blacksquare$

Definition. If V is a vector space and $A: V \to V$ is a linear map, and $\langle -,- \rangle_Q$ is an inner product on V, we define the adjoint of A to be the unique linear map $A^\dagger$ so that

$$\langle A\vec{v}, \vec{w} \rangle_Q = \langle \vec{v}, A^\dagger \vec{w} \rangle_Q$$

for all $\vec{v}, \vec{w}$ in V.

We say A is self-adjoint if $A = A^\dagger$.

Example. $V = \mathbb{R}^n$, $\langle -,- \rangle_Q = \langle -,- \rangle_{std}$

$\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^T \vec{w} \rangle$ for all $\vec{v}, \vec{w}$

so $A^T = A^\dagger$. In this case,

$$A \text{ is self-adjoint} \iff A \text{ is symmetric}$$

Example. $V = \mathbb{R}^n$, $\langle -,- \rangle_Q$ defined by $Q = Q^T$

$$\langle A\vec{v}, \vec{\omega} \rangle_Q = \langle A\vec{v}, Q\vec{\omega} \rangle = \langle \vec{v}, A^T Q \vec{\omega} \rangle$$

$$= \langle \vec{v}, Q Q^{-1} A^T Q \vec{\omega} \rangle$$

$$= \langle \vec{v}, Q^{-1} A^T Q \vec{\omega} \rangle_Q$$

In this case,

$$A^\dagger = Q^{-1} A^T Q.$$

In this case, A is self-adjoint $\iff$

$$A = A^\dagger = Q^{-1} A^T Q$$

or

$$QA = A^T Q.$$

Proposition. The shape operator is self-adjoint with respect to the I_p inner product.

Proof.

$$\langle S_p(\vec{v}), \vec{\omega} \rangle_{\mathrm{I}_p} = \langle (\mathrm{I}_p)^{-1} \mathrm{II}_p \vec{v}, \vec{\omega} \rangle_{\mathrm{I}_p}$$

$\mathrm{I}_p = \mathrm{I}_p^T$

$$= \langle (\mathrm{I}_p^T)^{-1} \mathrm{II}_p \vec{v}, \mathrm{I}_p \vec{\omega} \rangle_{\mathbb{R}^2}$$

$(A^{-1})^T = (A^T)^{-1}$

$$= \langle (\mathrm{I}_p^{-1})^T \mathrm{II}_p \vec{v}, \mathrm{I}_p \vec{\omega} \rangle_{\mathbb{R}^2}$$

$$= \langle \vec{v}, \mathrm{II}_p^T \overset{I}{\cancel{\mathrm{I}_p^{-1} \mathrm{I}_p}} \vec{\omega} \rangle_{\mathbb{R}^2}$$

$\mathrm{II}_p = \mathrm{II}_p^T$

$$= \langle \vec{v}, \underbrace{\mathrm{I}_p \mathrm{I}_p^{-1}}_{I} \mathrm{II}_p \vec{\omega} \rangle_{\mathbb{R}^2}$$

$$= \langle \vec{v}, \mathrm{I}_p^{-1} \mathrm{II}_p \vec{\omega} \rangle_{\mathrm{I}_p} = \langle \vec{v}, S_p(\vec{\omega}) \rangle_{\mathrm{I}_p}. \quad \blacksquare$$

(Weird) example. (sheared cylinder)

$$\vec{X}(u,v) = (\cos v, \sin v, u+v)$$

$$\vec{X}_u = (0,0,1)$$
$$\vec{X}_v = (-\sin v, \cos v, 1)$$
$$\Rightarrow E = 1 \quad F = 1 \quad G = 2$$

$$\vec{n} = (-\cos v, -\sin v, 0)$$
$$\vec{n}_u = (0,0,0)$$
$$\vec{n}_v = (\sin v, -\cos v, 0)$$
$$\Rightarrow \ell = 0 \quad m = 0 \quad n = 1$$

So

$$S_p = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

The shape operator is self-adjoint, but the matrix for the shape operator may not be symmetric.

We now need a theorem.

Theorem. [Roman, Advanced Linear Algebra, Thm 17.1] Suppose that V is an n-dimensional vector space with inner product $\langle -,- \rangle_V$ and $A: V \to V$ is self-adjoint. Then $\exists$ a $\langle -,- \rangle_V$ orthonormal basis $\vec{v}_1, \ldots, \vec{v}_n$ for V so that $A\vec{v}_i = \lambda_i v_i$ for real $\lambda_1, \ldots, \lambda_n$.

These are called <u>eigenvectors</u> and <u>eigenvalues</u> for the linear map A, and this theorem is generalization of the "symmetric implies diagonalizable" theorem from linear algebra class.

Since $S_p : T_pM \to T_pM$ is self-adjoint,

Definition. The eigenvectors of S_p are called the principal directions. The eigenvalues $\kappa_1 \geq \kappa_2$ are called the principal curvatures.

Lemma. If $\vec{v} = \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2$, then

$$\langle \vec{v}, S_p(\vec{v}) \rangle_{I_p} = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

and hence κ_1, κ_2 are the max and min value of $\langle \vec{v}, S_p(\vec{v}) \rangle_{I_p}$ over all (unit) $\vec{v}$.

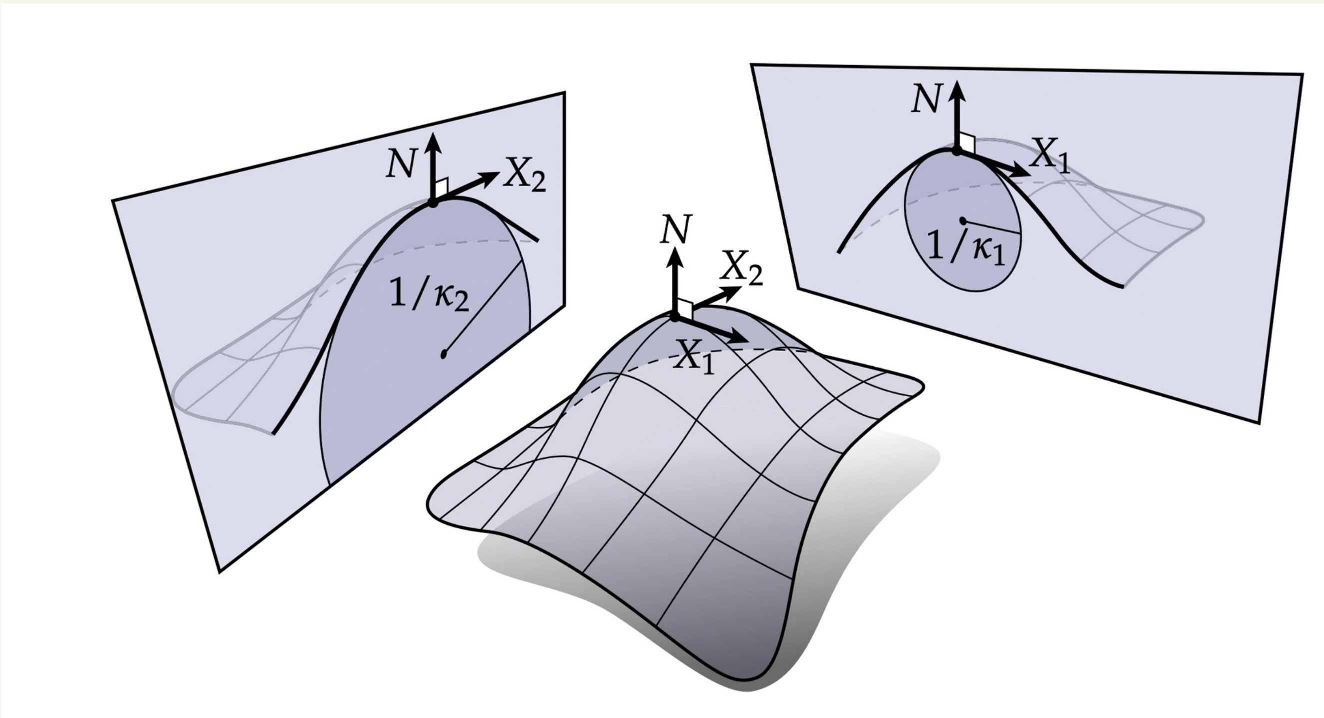
Proof. We compute

$$\langle \vec{v}, S_p(\vec{v}) \rangle = \langle \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2, \kappa_1 \cos\theta\, \vec{v}_1 + \kappa_2 \sin\theta\, \vec{v}_2 \rangle_{I_p}$$

(FOIL and use orthonormality)

$$= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta. \qquad \blacksquare$$

Now that we have a handle on $S_{\vec{p}}$, we can start to interpret it geometrically.



Theorem. If $\vec{v} \in T_{\vec{p}}M$, then $\vec{v}, \vec{n}$ span a plane P through $\vec{p}$. Let $\vec{\alpha}$ be the plane curve $P \cap M$. Then the signed curvature of $\vec{\alpha}$ at $\vec{p}$

$$K_{\pm} = \frac{\langle \vec{v}, S_p(\vec{v}) \rangle_{I_p}}{\langle \vec{v}, \vec{v} \rangle_{I_p}}$$

Proof. Parametrize $\vec{\alpha}$ so that

$\vec{\alpha}(t) = \vec{X}(\vec{\beta}(t))$, $\vec{\alpha}(0) = \vec{p}$, $\vec{\alpha}'(0) = \vec{v}$.

(where $\vec{\beta} : \mathbb{R} \to \mathbb{R}^2$, the u-v plane)

We now compute

$$K_{\pm}(0) = \frac{\langle \vec{\alpha}''(0), \vec{\alpha}'(0)^{\perp} \rangle}{\langle \vec{\alpha}'(0), \vec{\alpha}'(0) \rangle}$$

Since $\vec{\alpha}'(0) = \vec{v}$ and we have oriented the plane by $\vec{v}, \vec{n}$, $\vec{\alpha}'(0)^{\perp} = \vec{n} = \vec{g}(\vec{\alpha}(0))$.

$$= \frac{\langle \vec{\alpha}''(0), \vec{g}(\vec{\alpha}(0)) \rangle}{\langle \vec{v}, \vec{v} \rangle}$$

where $\vec{g} : M \to S^2$ is the Gauss map. Now $\vec{\alpha}(t) \in M$, so $\vec{\alpha}'(t) \in T_pM$, so

$$\langle \vec{\alpha}'(t), \vec{g}(\vec{\alpha}(t)) \rangle \equiv 0$$

Differentiating w.r.t. t,

$$0 = \frac{d}{dt}\left\langle \vec{\alpha}'(t), \vec{g}(\vec{\alpha}(t)) \right\rangle$$

$$= \left\langle \vec{\alpha}''(t), \vec{g}(\vec{\alpha}(t)) \right\rangle + \left\langle \vec{\alpha}'(t), D\vec{g}(\vec{\alpha}(t))\vec{\alpha}'(t) \right\rangle$$

Evaluating at 0,

$$\left\langle \vec{\alpha}''(0), \vec{n} \right\rangle = -\left\langle \vec{v}, D\vec{g}(\vec{p})\,\vec{v} \right\rangle$$

$$= \left\langle \vec{v}, S_p(\vec{v}) \right\rangle \quad \blacksquare$$

It follows that κ_1, κ_2 are two such curvatures.

Definition. The bilinear form II_p

$\left\langle \vec{v}, \vec{w} \right\rangle_{\mathrm{II}_p} = \left\langle \vec{v}, S_p(\vec{w}) \right\rangle_{\mathrm{I}_p}$ is called

the <u>second</u> <u>fundamental</u> <u>form</u>.

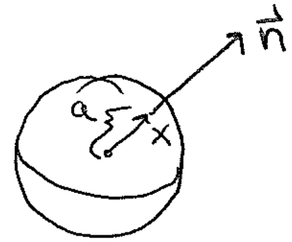
Examples.

If $S_p(\vec{v}) = 0$ for all p, v then M is a plane (or a subset of a plane).

Since $S_p(\vec{v}) = -D_{\vec{v}}\vec{n}$, this shows all directional derivatives of $\vec{n}$ are 0. Hence $\vec{n}$ is constant and M can't leave the plane normal to $\vec{n}$.

If M is a sphere, of radius a, then $S_p(\vec{v}) = -\frac{1}{a}\vec{v}$.

We know that



$\vec{n}$ is a unit vector in the direction of $\vec{x}$, so $\vec{n} = \frac{1}{a}\vec{x}$. Thus $D_{\vec{v}}\vec{n} = D_{\vec{v}}(\frac{1}{a}\vec{x}) = \frac{1}{a}\vec{v}$. This last takes a little unpacking, but recall that x is a function of u and v (the parameters) so

$$\vec{n}(u,v) = \frac{1}{a}\vec{X}(u,v) \Rightarrow \begin{aligned} \vec{n}_u &= \frac{1}{a}\vec{X}_u \\ \vec{n}_v &= \frac{1}{a}\vec{X}_v \end{aligned}$$

Thus if $\vec{V} = \lambda_1 \vec{X}_u + \lambda_2 \vec{X}_v$ we have

$$D_{\vec{V}}\vec{n} = \lambda_1 \cancel{x}n_u + \lambda_2 n_v =$$

$$= \cancel{\tfrac{1}{a}}\, \lambda_1/a\, X_u + \lambda_2/a\, X_v$$

$$= \frac{1}{a}(\lambda_1 X_u + \lambda_2 X_v) = \frac{1}{a}\vec{V}$$

as claimed. Thus $S_p(\vec{v}) = -D_{\vec{v}}n = -\frac{1}{a}\vec{v}$.