The Tractrix

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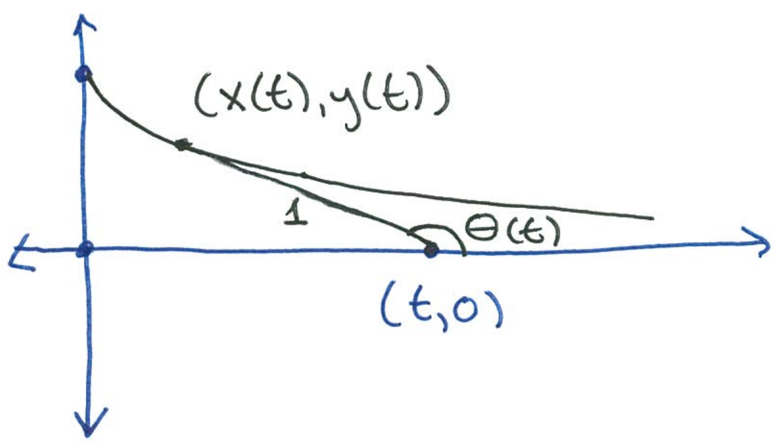
A mass is located at $(0,1)$ and pulled by a linkage of fixed length 1 moving along the x-axis at speed 1.

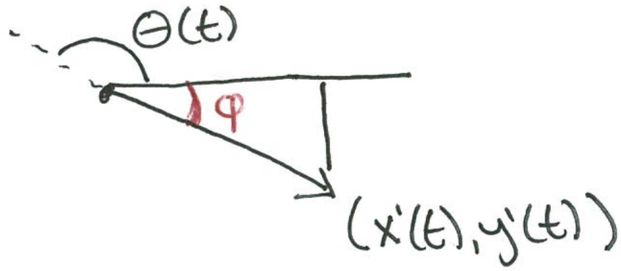


We know

$$x(t) = t + \cos\theta$$
$$y(t) = \sin\theta$$

because of the length-1 constraint.

Less obviously, the linkage is tangent to the curve, so we know that we have a triangle



Since

recall $y'(t)$ is negative!

$$\tan\varphi = \frac{-y'(t)}{x'(t)}$$

and $\varphi = \pi - \theta$, the supplementary angle formula for tan tells us that

$$\tan\theta = \frac{y'(t)}{x'(t)} = \frac{\cos\theta\ \theta'(t)}{1 - \sin\theta\ \theta'(t)}$$

We can solve this formula for $\theta'(t)$.

$$\tan\theta\,(1 - \sin\theta\,\theta') = \cos\theta\,\theta'$$

$$\tan\theta - \tan\theta\sin\theta\,\theta' = \cos\theta\,\theta'$$

$$\tan\theta = \left(\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right)\theta'$$

→ multiplying through by cos.

$$\sin\theta = \theta'$$

We can solve this by separation of variables: $\sin\theta = \frac{d\theta}{dt}$, so

$$\int \frac{1}{\sin\theta}\,d\theta = \int 1\,dt$$

~~and~~ or

$$\int \csc\theta\,d\theta = -\ln(\csc\theta + \cot\theta) + C$$

$$= t$$

for some constant c.

$$\tan\theta\,(1-\sin\theta\cdot\theta') = \cos\theta\cdot\theta'$$

$$\tan\theta \;-\; \tan\theta\sin\theta\;\theta' = \cos\theta\cdot\theta'$$

$$\tan\theta = (\cos\theta + \tan\theta\sin\theta)\cdot\theta'$$

$$\frac{\sin\theta}{\cos\theta} = \left(\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right)\cdot\theta'$$

Now multiply by $\cos\theta$ to get

$$\sin\theta = (\cos^2\theta + \sin^2\theta)\cdot\theta' = \theta'$$

This is the first differential equation we've seen in a while. So let's solve carefully:

$$\sin\theta(t) = \theta'(t)$$

$$1 = \frac{1}{\sin\theta(t)}\cdot\theta'(t)$$

Now integrating both sides wrt t,

$$\int 1\,dt = \int \csc\theta(t)\cdot\theta'(t)\,dt$$

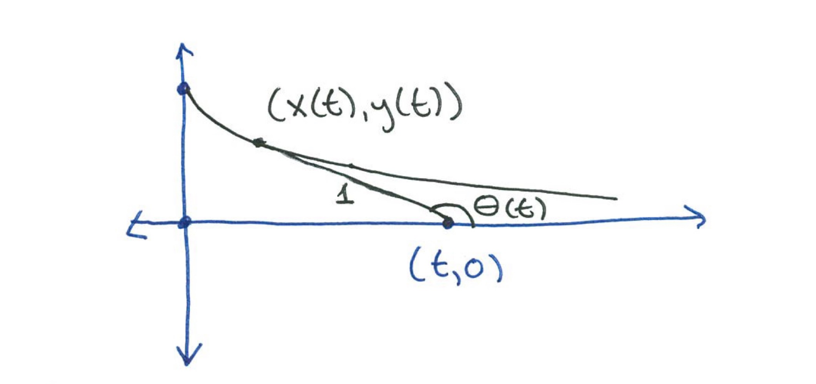
u-substitution

$$= \int \csc\theta\,d\theta$$

look it up

$$= -\ln(\csc\theta + \cot\theta)$$

So $t + C = -\ln(\csc\theta + \cot\theta)$ for some C.



At $t=0$, $\theta(0) = \pi/2$. So

$$C = -\ln(\csc\pi/2 + \cot\pi/2) = -\ln 1 = 0.$$

So

$$t = -\ln(\csc\theta + \cot\theta)$$

We can simplify the right hand side:

$$\csc\theta + \cot\theta = \frac{1+\cos\theta}{\sin\theta}$$

Now $\frac{1+\cos\theta}{2} = \cos^2\theta/2$ (half-angle), so we try writing everything in terms of $\theta/2$. $\sin\theta = 2\cos\theta/2 \sin\theta/2$, so

$$\frac{1+\cos\theta}{\sin\theta} = \frac{\cancel{2}\cos^{\cancel{2}}\theta/2}{\cancel{2}\cancel{\cos\theta/2}\sin\theta/2} = \cot\theta/2.$$

and

laws of logs

$$t = -\ln\cot\theta/2 = \ln\tan\theta/2$$

Plugging back into

$$x(t) = t + \cos\theta$$
$$y(t) = \sin\theta$$

we get a parametrization in θ:

$$x(\theta) = \ln\tan\theta/2 + \cos\theta$$
$$y(\theta) = \sin\theta$$

What if we want a parametrization in t instead of θ? Well,

$$t = \ln\tan\theta/2$$
$$e^t = \tan\theta/2$$

so we must write $\cos\theta$, $\sin\theta$ in terms of $\tan\theta/2$.

This is a great time to look at the Wikipedia page on trig identities:

$$\sin(2\theta) = 2\sin\theta\cos\theta = (\sin\theta + \cos\theta)^2 - 1 = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

So we get

$$\cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} = \frac{1-e^{2t}}{1+e^{2t}}$$

$$\sin\theta = \frac{2\tan\theta/2}{1+\tan\theta/2} = \frac{2e^t}{1+e^{2t}}$$

This is a perfectly good parametrization

$$x(t) = t + \frac{1-e^{2t}}{1+e^{2t}}$$

$$y(t) = \frac{2e^t}{1+e^{2t}}$$

But we can make it better! Let

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

These are the hyperbolic trig functions. You'll prove in homework that

$$\frac{d}{dt} \sinh t = \cosh t$$

$$\frac{d}{dt} \cosh t = + \sinh t$$

and $\cosh^2 t - \sinh^2 t = 1$. Now

$$\frac{1 - e^{2t}}{1 + e^{2t}} \cdot \frac{e^{-t}}{e^{-t}} = \frac{e^{-t} - e^{t}}{e^{-t} + e^{t}} = \frac{-\sinh t}{\cosh t} = -\tanh t$$

and

$$\frac{2e^t}{1+e^{2t}} \cdot \frac{e^{-t}}{e^{-t}} = \frac{2}{e^{-t}+e^t} = \operatorname{sech} t$$

so we have a very simple parametrization

$$x(t) = t - \tanh t$$
$$y(t) = \operatorname{sech} t$$

for $t \geq 0$.

The tractrix will be important to us later in the course!