usual, work with a parametrization where the u-curves are lines of curvature with principal curvature k_1 and the v-curves are lines of curvature with principal curvature k_2. Use Lemma 3.3 to show that the u-curves have curvature $|k_1|$ and are planar. Then define $\boldsymbol{\alpha}$ appropriately and check that it is a regular curve.)

17. If M is a surface with both principal curvatures constant, prove that M is (a subset of) either a sphere, a plane, or a right circular cylinder. (Hint: See Exercise 2.2.14, Proposition 3.4, and Exercise 16.)

18. Consider the parametrized surfaces

$$\mathbf{x}(u, v) = (-\cosh u \sin v, \cosh u \cos v, u) \qquad \text{(a catenoid)}$$
$$\mathbf{y}(u, v) = (u \cos v, u \sin v, v) \qquad \text{(a helicoid)}.$$

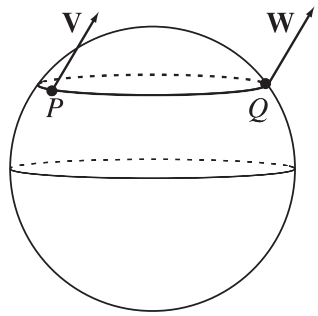
   a. Compute the first and second fundamental forms of both surfaces, and check that both surfaces are minimal.
   b. Find the asymptotic curves on both surfaces.
   c. Show that we can locally reparametrize the helicoid in such a way as to make the first fundamental forms of the two surfaces agree; this means that the two surfaces are locally isometric. (Hint: See p. 39. Replace u with $\sinh u$ in the parametrization of the helicoid. Why is this legitimate?)
   d. Why are they not globally isometric?
   e. (for the student who's seen a bit of complex variables) As a hint to what's going on here, let $z = u + iv$ and $\mathbf{Z} = \mathbf{x} + i\mathbf{y}$, and check that, continuing to use the substitution from part c, $\mathbf{Z} = (\sin iz, \cos iz, z)$. Understand now how one can obtain a one-parameter family of isometric surfaces interpolating between the helicoid and the catenoid.

19. Find all the surfaces of revolution of constant curvature
   a. $K = 0$
   b. $K = 1$
   c. $K = -1$

   (Hint: There are more than you might suspect. But your answers will involve integrals you cannot express in terms of elementary functions.)

4. Covariant Differentiation, Parallel Translation, and Geodesics

Now we turn to the "intrinsic" geometry of a surface, i.e., the geometry that can be observed by an inhabitant (for example, a very thin ant) of the surface, who can only perceive what happens along (or, say, tangential to) the surface. Anyone who has studied Euclidean geometry knows how important the notion of *parallelism* is (and classical non-Euclidean geometry arises when one removes Euclid's parallel postulate, which stipulates that given any line L in the plane and any point P not lying on L, there is a unique line through P parallel to L). It seems quite intuitive to say that, working just in $\mathbb{R}^3$, two vectors $\mathbf{V}$ (thought of as being "tangent at P") and $\mathbf{W}$ (thought of as being "tangent at Q") are parallel provided that we obtain $\mathbf{W}$ when we move $\mathbf{V}$ "parallel to itself" from P to Q; in other words, if $\mathbf{W} = \mathbf{V}$. But what would an inhabitant of the sphere say? How should he compare a tangent vector at one point of the sphere to a tangent vector



Are **V** and **W** parallel?

FIGURE 4.1

at another and determine if they're "parallel"? (See Figure 4.1.) Perhaps a better question is this: Given a curve $\boldsymbol{\alpha}$ on the surface and a vector field $\mathbf{X}$ defined along $\boldsymbol{\alpha}$, should we say $\mathbf{X}$ is parallel if it has zero derivative along $\boldsymbol{\alpha}$?

We already know how an inhabitant differentiates a scalar function $f: M \to \mathbb{R}$, by considering the directional derivative $D_{\mathbf{V}} f$ for any tangent vector $\mathbf{V} \in T_P M$. We now begin with a

Definition. We say a function $\mathbf{X}: M \to \mathbb{R}^3$ is a *vector field* on M if

(1) $\mathbf{X}(P) \in T_P M$ for every $P \in M$, and

(2) for any parametrization $\mathbf{x}: U \to M$, the function $\mathbf{X} \circ \mathbf{x}: U \to \mathbb{R}^3$ is (continuously) differentiable.

Now, we can differentiate a vector field $\mathbf{X}$ on M in the customary fashion: If $\mathbf{V} \in T_P M$, we choose a curve $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$ and set $D_{\mathbf{V}}\mathbf{X} = (\mathbf{X} \circ \boldsymbol{\alpha})'(0)$. (As usual, the chain rule tells us this is well-defined.) But the inhabitant of the surface can only see that portion of this vector lying in the tangent plane. This brings us to the

Definition. Given a vector field $\mathbf{X}$ and $\mathbf{V} \in T_P M$, we define the *covariant derivative*

$$\begin{aligned}\nabla_{\mathbf{V}}\mathbf{X} = (D_{\mathbf{V}}\mathbf{X})^{\parallel} &= \text{the projection of } D_{\mathbf{V}}\mathbf{X} \text{ onto } T_P M \\ &= D_{\mathbf{V}}\mathbf{X} - (D_{\mathbf{V}}\mathbf{X} \cdot \mathbf{n})\mathbf{n}.\end{aligned}$$

Given a curve $\boldsymbol{\alpha}$ in M, we say the vector field $\mathbf{X}$ is *covariant constant* or *parallel* along $\boldsymbol{\alpha}$ if $\nabla_{\boldsymbol{\alpha}'(t)}\mathbf{X} = \mathbf{0}$ for all t. (This means that $D_{\boldsymbol{\alpha}'(t)}\mathbf{X} = (\mathbf{X} \circ \boldsymbol{\alpha})'(t)$ is a multiple of the normal vector $\mathbf{n}(\boldsymbol{\alpha}(t))$.)

Example 1. Let M be a sphere and let $\boldsymbol{\alpha}$ be a great circle in M. The derivative of the unit tangent vector of $\boldsymbol{\alpha}$ points towards the center of the circle, which is in this case the center of the sphere, and thus is completely normal to the sphere. Therefore, the unit tangent vector field of $\boldsymbol{\alpha}$ is parallel along $\boldsymbol{\alpha}$. Observe that the constant vector field $(0, 0, 1)$ *is* parallel along the equator $z = 0$ of a sphere centered at the origin. Is this true of any other constant vector field? ∇

Example 2. A fundamental example requires that we revisit the Christoffel symbols. Given a parametrized surface $\mathbf{x}: U \to M$, we have

$$\begin{aligned}\nabla_{\mathbf{x}_u}\mathbf{x}_u &= (\mathbf{x}_{uu})^{\parallel} = \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v \\ \nabla_{\mathbf{x}_v}\mathbf{x}_u &= (\mathbf{x}_{uv})^{\parallel} = \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v = \nabla_{\mathbf{x}_u}\mathbf{x}_v, \quad \text{and}\end{aligned}$$

$$\nabla_{\mathbf{x}_v}\mathbf{x}_v = (\mathbf{x}_{vv})^{\parallel} = \Gamma_{vv}^{u}\mathbf{x}_u + \Gamma_{vv}^{v}\mathbf{x}_v. \quad \nabla$$

The first result we prove is the following

Proposition 4.1. *Let I be an interval in $\mathbb{R}$ with $0 \in I$. Given a curve $\boldsymbol{\alpha}: I \to M$ with $\boldsymbol{\alpha}(0) = P$ and $\mathbf{X}_0 \in T_P M$, there is a unique parallel vector field $\mathbf{X}$ defined along $\boldsymbol{\alpha}$ with $\mathbf{X}(P) = \mathbf{X}_0$.*

Proof. Assuming $\boldsymbol{\alpha}$ lies in a parametrized portion $\mathbf{x}: U \to M$, set $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ and write $\mathbf{X}(\boldsymbol{\alpha}(t)) = a(t)\mathbf{x}_u(u(t), v(t)) + b(t)\mathbf{x}_v(u(t), v(t))$. Then $\boldsymbol{\alpha}'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$ (where the the cumbersome argument $(u(t), v(t))$ is understood). So, by the product rule and chain rule, we have

$$\begin{aligned}
\nabla_{\boldsymbol{\alpha}'(t)}\mathbf{X} &= \big((\mathbf{X}\circ\boldsymbol{\alpha})'(t)\big)^{\parallel} = \left(\frac{d}{dt}\big(a(t)\mathbf{x}_u(u(t), v(t)) + b(t)\mathbf{x}_v(u(t), v(t))\big)\right)^{\parallel} \\
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\left(\frac{d}{dt}\mathbf{x}_u(u(t), v(t))\right)^{\parallel} + b(t)\left(\frac{d}{dt}\mathbf{x}_v(u(t), v(t))\right)^{\parallel} \\
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\big(u'(t)\mathbf{x}_{uu} + v'(t)\mathbf{x}_{uv}\big)^{\parallel} + b(t)\big(u'(t)\mathbf{x}_{vu} + v'(t)\mathbf{x}_{vv}\big)^{\parallel} \\
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\big(u'(t)(\Gamma_{uu}^{u}\mathbf{x}_u + \Gamma_{uu}^{v}\mathbf{x}_v) + v'(t)(\Gamma_{uv}^{u}\mathbf{x}_u + \Gamma_{uv}^{v}\mathbf{x}_v)\big) \\
&\qquad + b(t)\big(u'(t)(\Gamma_{vu}^{u}\mathbf{x}_u + \Gamma_{vu}^{v}\mathbf{x}_v) + v'(t)(\Gamma_{vv}^{u}\mathbf{x}_u + \Gamma_{vv}^{v}\mathbf{x}_v)\big) \\
&= \big(a'(t) + a(t)(\Gamma_{uu}^{u}u'(t) + \Gamma_{uv}^{u}v'(t)) + b(t)(\Gamma_{vu}^{u}u'(t) + \Gamma_{vv}^{u}v'(t))\big)\mathbf{x}_u \\
&\qquad + \big(b'(t) + a(t)(\Gamma_{uu}^{v}u'(t) + \Gamma_{uv}^{v}v'(t)) + b(t)(\Gamma_{vu}^{v}u'(t) + \Gamma_{vv}^{v}v'(t))\big)\mathbf{x}_v.
\end{aligned}$$

Thus, to say $\mathbf{X}$ is parallel along the curve $\boldsymbol{\alpha}$ is to say that $a(t)$ and $b(t)$ are solutions of the linear system of first order differential equations

$$(\clubsuit) \qquad \begin{aligned}
a'(t) + a(t)(\Gamma_{uu}^{u}u'(t) + \Gamma_{uv}^{u}v'(t)) + b(t)(\Gamma_{vu}^{u}u'(t) + \Gamma_{vv}^{u}v'(t)) &= 0 \\
b'(t) + a(t)(\Gamma_{uu}^{v}u'(t) + \Gamma_{uv}^{v}v'(t)) + b(t)(\Gamma_{vu}^{v}u'(t) + \Gamma_{vv}^{v}v'(t)) &= 0.
\end{aligned}$$

By Theorem 3.2 of the Appendix, this system has a unique solution on I once we specify $a(0)$ and $b(0)$, and hence we obtain a unique parallel vector field $\mathbf{X}$ with $\mathbf{X}(P) = \mathbf{X}_0$. $\square$

Definition. If $\boldsymbol{\alpha}$ is a path from P to Q, we refer to $\mathbf{X}(Q)$ as the *parallel translate* of $\mathbf{X}(P) = \mathbf{X}_0 \in T_P M$ along $\boldsymbol{\alpha}$, or the result of *parallel translation* along $\boldsymbol{\alpha}$.

Remark. The system of differential equations ($\clubsuit$) that defines parallel translation shows that it is "intrinsic," i.e., depends only on the first fundamental form of M, despite our original extrinsic definition. In particular, parallel translation in locally isometric surfaces will be identical.

Example 3. Fix a latitude circle $u = u_0$ ($u_0 \neq 0, \pi$) on the unit sphere (see Example 1(d) on p. 37) and let's calculate the effect of parallel-translating the vector $\mathbf{X}_0 = \mathbf{x}_v$ starting at the point P given by $u = u_0$, $v = 0$, once around the circle, counterclockwise. We parametrize the curve by $u(t) = u_0$, $v(t) = t$, $0 \le t \le 2\pi$. Using our computation of the Christoffel symbols of the sphere in Example 1 or 2 of Section 3, we obtain from ($\clubsuit$) the differential equations

$$\begin{aligned}
a'(t) &= \sin u_0 \cos u_0 b(t), & a(0) &= 0 \\
b'(t) &= -\cot u_0 a(t), & b(0) &= 1.
\end{aligned}$$

We solve this system by differentiating the second equation again and substituting the first:

$$b''(t) = -\cot u_0 a'(t) = -\cos^2 u_0 b(t), \qquad b(0) = 1.$$

Recalling that every solution of the differential equation $y''(t) + k^2 y(t) = 0$ is of the form $y(t) = c_1 \cos(kt) + c_2 \sin(kt)$, $c_1, c_2 \in \mathbb{R}$, we see that the solution is

$$a(t) = \sin u_0 \sin\big((\cos u_0)t\big), \qquad b(t) = \cos\big((\cos u_0)t\big).$$

Note that $\|\mathbf{X}(\boldsymbol{\alpha}(t))\|^2 = Ea(t)^2 + 2Fa(t)b(t) + Gb(t)^2 = \sin^2 u_0$ for all t. That is, the original vector $\mathbf{X}_0$ rotates as we parallel translate it around the latitude circle, and its length is preserved. As we see in Figure 4.2, the vector rotates clockwise as we proceed around the latitude circle (in the upper hemisphere). But

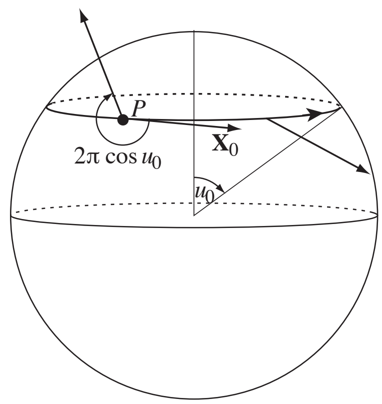


FIGURE 4.2

this makes sense: If we just take the covariant derivative of the vector field tangent to the circle, it points upwards (cf. Figure 3.1), so the vector field must rotate clockwise to counteract that effect in order to remain parallel. Since $b(2\pi) = \cos(2\pi \cos u_0)$, we see that the vector turns through an angle of $-2\pi \cos u_0$. ∇

Example 4 (Foucault pendulum)**.** Foucault observed in 1851 that the swing plane of a pendulum located on the latitude circle $u = u_0$ precesses with a period of $T = 24/\cos u_0$ hours. We can use the result of Example 3 to explain this. We imagine the earth as fixed and "transport" the swinging pendulum once around the circle in 24 hours. If we make the pendulum very long and the swing rather short, the motion will be "essentially" tangential to the surface of the earth. If we move slowly around the circle, the forces will be "essentially" normal to the sphere: In particular, letting R denote the radius of the earth (approximately 3960 mi), the tangential component of the centripetal acceleration is (cf. Figure 3.1)

$$(R \sin u_0) \cos u_0 \left(\frac{2\pi}{24}\right)^2 \leq \frac{2\pi^2 R}{24^2} \approx 135.7 \text{ mi/hr}^2 \approx 0.0553 \text{ ft/sec}^2 \approx 0.17\% g.$$

Thus, the "swing vector field" is, for all practical purposes, parallel along the curve. Therefore, it turns through an angle of $2\pi \cos u_0$ in one trip around the circle, so it takes $\dfrac{2\pi}{(2\pi \cos u_0)/24} = \dfrac{24}{\cos u_0}$ hours to return to its original swing plane. ∇

Our experience in Example 3 suggests the following

Proposition 4.2. *Parallel translation preserves lengths and angles. That is, if* $\mathbf{X}$ *and* $\mathbf{Y}$ *are parallel vector fields along a curve* $\boldsymbol{\alpha}$ *from* P *to* Q, *then* $\|\mathbf{X}(P)\| = \|\mathbf{X}(Q)\|$ *and the angle between* $\mathbf{X}(P)$ *and* $\mathbf{Y}(P)$ *equals the angle between* $\mathbf{X}(Q)$ *and* $\mathbf{Y}(Q)$ *(assuming these are nonzero vectors).*

Proof. Consider $f(t) = \mathbf{X}(\boldsymbol{\alpha}(t)) \cdot \mathbf{Y}(\boldsymbol{\alpha}(t))$. Then

$$\begin{aligned} f'(t) &= (\mathbf{X}\circ\boldsymbol{\alpha})'(t) \cdot (\mathbf{Y}\circ\boldsymbol{\alpha})(t) + (\mathbf{X}\circ\boldsymbol{\alpha})(t) \cdot (\mathbf{Y}\circ\boldsymbol{\alpha})'(t) \\ &= D_{\boldsymbol{\alpha}'(t)}\mathbf{X} \cdot \mathbf{Y} + \mathbf{X} \cdot D_{\boldsymbol{\alpha}'(t)}\mathbf{Y} \overset{(1)}{=} \nabla_{\boldsymbol{\alpha}'(t)}\mathbf{X} \cdot \mathbf{Y} + \mathbf{X} \cdot \nabla_{\boldsymbol{\alpha}'(t)}\mathbf{Y} \overset{(2)}{=} 0. \end{aligned}$$

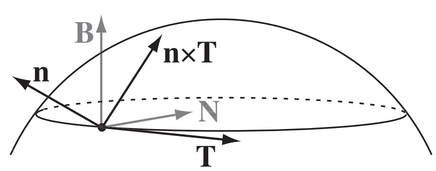
Note that equality (1) holds because $\mathbf{X}$ and $\mathbf{Y}$ are tangent to M and hence their dot product with any vector normal to the surface is 0. Equality (2) holds because $\mathbf{X}$ and $\mathbf{Y}$ are assumed parallel along $\boldsymbol{\alpha}$. It follows that the dot product $\mathbf{X} \cdot \mathbf{Y}$ remains constant along $\boldsymbol{\alpha}$. Taking $\mathbf{Y} = \mathbf{X}$, we infer that $\|\mathbf{X}\|$ (and similarly $\|\mathbf{Y}\|$) is constant. Knowing that, using the famous formula $\cos\theta = \mathbf{X} \cdot \mathbf{Y}/\|\mathbf{X}\|\|\mathbf{Y}\|$ for the angle θ between $\mathbf{X}$ and $\mathbf{Y}$, we infer that the angle remains constant. ☐

Now we change gears somewhat. We saw in Exercise 1.1.8 that the shortest path joining two points in $\mathbb{R}^3$ is a line segment and in Exercise 1.3.1 that the shortest path joining two points on the unit sphere is a great circle. One characterization of the line segment is that it never changes direction, so that its unit tangent vector is parallel (so no distance is wasted by turning). (What about the sphere?) It seems plausible that the mythical inhabitant of our general surface M might try to travel from one point to another in M, *staying in* M, by similarly not turning; that is, so that his unit tangent vector field is parallel along his path. Physically, this means that if he travels at constant speed, any acceleration should be normal to the surface. This leads us to the following

Definition. We say a parametrized curve $\boldsymbol{\alpha}$ in a surface M is a *geodesic* if its tangent vector is parallel along the curve, i.e., if $\nabla_{\boldsymbol{\alpha}'}\boldsymbol{\alpha}' = 0$.

Recall that since parallel translation preserves lengths, $\boldsymbol{\alpha}$ must have constant speed, although it may not be arclength-parametrized. In general, we refer to an unparametrized curve as a geodesic if its arclength parametrization is in fact a geodesic.

In general, given any arclength-parametrized curve $\boldsymbol{\alpha}$ lying on M, we defined its normal curvature at the end of Section 2. Instead of using the Frenet frame, it is natural to consider the *Darboux frame* for $\boldsymbol{\alpha}$, which takes into account the fact that $\boldsymbol{\alpha}$ lies on the surface M. (Both are illustrated in Figure 4.3.) We take



The Frenet and Darboux frames

FIGURE 4.3

the right-handed orthonormal basis $\{\mathbf{T}, \mathbf{n} \times \mathbf{T}, \mathbf{n}\}$; note that the first two vectors give a basis for $T_P M$. We can decompose the curvature vector

$$\kappa\mathbf{N} = \big(\underbrace{\kappa\mathbf{N} \cdot (\mathbf{n} \times \mathbf{T})}_{\kappa_g}\big)(\mathbf{n} \times \mathbf{T}) + \big(\underbrace{\kappa\mathbf{N} \cdot \mathbf{n}}_{\kappa_n}\big)\mathbf{n}.$$

As we saw before, κ_n gives the *normal* component of the curvature vector; κ_g gives the *tangential* component of the curvature vector and is called the *geodesic curvature*. This terminology arises from the fact that $\boldsymbol{\alpha}$ is a geodesic if and only if its geodesic curvature vanishes. (When $\kappa = 0$, the principal normal is not defined, and we really should write $\boldsymbol{\alpha}''$ in the place of $\kappa\mathbf{N}$. If the acceleration vanishes at a point, then certainly its normal and tangential components are both $\mathbf{0}$.)

Example 5. We saw in Example 1 that every great circle on a sphere is a geodesic. Are there others? Let $\boldsymbol{\alpha}$ be a geodesic on a sphere centered at the origin. Since $\kappa_g = 0$, the acceleration vector $\boldsymbol{\alpha}''(s)$ must be a multiple of $\boldsymbol{\alpha}(s)$ for every s, and so $\boldsymbol{\alpha}'' \times \boldsymbol{\alpha} = \mathbf{0}$. Therefore $\boldsymbol{\alpha}' \times \boldsymbol{\alpha} = \mathbf{A}$ is a constant vector, so $\boldsymbol{\alpha}$ lies in the plane passing through the origin with normal vector $\mathbf{A}$. That is, $\boldsymbol{\alpha}$ is a great circle. ∇

Remark. We saw in Example 3 that a vector rotates clockwise at a constant rate as we parallel translate along the latitude circle of the sphere. If we think about the unit tangent vector $\mathbf{T}$ moving counterclockwise along this curve, its covariant derivative along the curve points up the sphere, as shown in Figure 4.4, i.e., "to the left." Thus, we must compensate by steering "to the right" in order to have no net turning (i.e., to

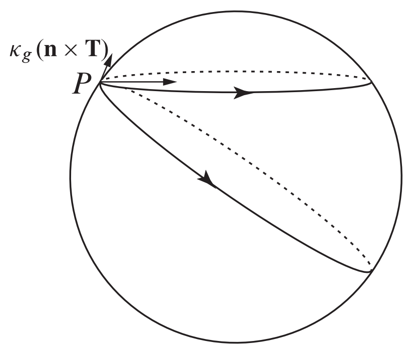


FIGURE 4.4

make the covariant derivative zero). Of course, this makes sense also because, according to Example 5, the geodesic that passes through P in the same direction heads "downhill," to the right.

Using the equations (♣), let's now give the equations for the curve $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ to be a geodesic. Since $\mathbf{X} = \boldsymbol{\alpha}'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$, we have $a(t) = u'(t)$ and $b(t) = v'(t)$, and the resulting equations are

(♣♣)
$$\begin{aligned} u''(t) + \Gamma^u_{uu}u'(t)^2 + 2\Gamma^u_{uv}u'(t)v'(t) + \Gamma^u_{vv}v'(t)^2 &= 0 \\ v''(t) + \Gamma^v_{uu}u'(t)^2 + 2\Gamma^v_{uv}u'(t)v'(t) + \Gamma^v_{vv}v'(t)^2 &= 0. \end{aligned}$$

The following result is a consequence of basic results on differential equations (see Theorem 3.1 of the Appendix).

Proposition 4.3. *Given a point $P \in M$ and $\mathbf{V} \in T_PM$, $\mathbf{V} \neq \mathbf{0}$, there exist $\varepsilon > 0$ and a* unique *geodesic $\boldsymbol{\alpha}\colon (-\varepsilon, \varepsilon) \to M$ with $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$.*

Example 6. We now use the equations (♣♣) to solve for geodesics analytically in a few examples.

(a) Let $\mathbf{x}(u,v) = (u,v)$ be the obvious parametrization of the plane. Then all the Christoffel symbols vanish and the geodesics are the solutions of

$$u''(t) = v''(t) = 0,$$

so we get the lines $\boldsymbol{\alpha}(t) = (u(t), v(t)) = (a_1 t + b_1, a_2 t + b_2)$, as expected. Note that $\boldsymbol{\alpha}$ does in fact have constant speed.

(b) Using the standard spherical coordinate parametrization of the sphere, we obtain (see Example 1 or 2 of Section 3) the equations

$$(*) \qquad u''(t) - \sin u(t) \cos u(t) v'(t)^2 = 0 = v''(t) + 2\cot u(t) u'(t) v'(t).$$

Well, one obvious set of solutions is to take $u(t) = t$, $v(t) = v_0$ (and these, indeed, give the great circles through the north pole). Integrating the second equation in $(*)$ we obtain $\ln v'(t) = -2\ln\sin u(t) + \text{const}$, so

$$v'(t) = \frac{c}{\sin^2 u(t)}$$

for some constant c. Substituting this in the first equation in $(*)$ we find that

$$u''(t) - \frac{c^2 \cos u(t)}{\sin^3 u(t)} = 0;$$

multiplying both sides by $u'(t)$ (the "energy trick" from physics) and integrating, we get

$$u'(t)^2 = C^2 - \frac{c^2}{\sin^2 u(t)}, \quad \text{and so} \quad u'(t) = \pm\sqrt{C^2 - \frac{c^2}{\sin^2 u(t)}}$$

for some constant C. Switching to Leibniz notation for obvious reasons, we obtain

$$\frac{dv}{du} = \frac{v'(t)}{u'(t)} = \pm\frac{c\csc^2 u}{\sqrt{C^2 - c^2\csc^2 u}}; \quad \text{thus, separating variables gives}$$
$$dv = \pm\frac{c\csc^2 u\,du}{\sqrt{C^2 - c^2\csc^2 u}} = \pm\frac{c\csc^2 u\,du}{\sqrt{(C^2 - c^2) - c^2\cot^2 u}}.$$

Now we make the substitution $c\cot u = \sqrt{C^2 - c^2}\sin w$; then we have

$$dv = \pm\frac{c\csc^2 u\,du}{\sqrt{(C^2 - c^2) - c^2\cot^2 u}} = \mp dw,$$

and so, at long last, we have $w = \pm v + a$ for some constant a. Thus,

$$c\cot u = \sqrt{C^2 - c^2}\sin w = \sqrt{C^2 - c^2}\sin(\pm v + a) = \sqrt{C^2 - c^2}(\sin a\cos v \pm \cos a\sin v),$$

and so, finally, we have the equation

$$c\cos u + \sqrt{C^2 - c^2}\sin u(A\cos v + B\sin v) = 0,$$

which we should recognize as the equation of a great circle! (Here's a hint: This curve lies on the plane $\sqrt{C^2 - c^2}(Ax + By) + cz = 0$.) ∇

We can now give a beautiful geometric description of the geodesics on a surface of revolution.

Proposition 4.4 (Clairaut's relation). *The geodesics on a surface of revolution satisfy the equation*

$$(\diamondsuit) \qquad r\cos\phi = \text{const},$$

where r is the distance from the axis of revolution and ϕ is the angle between the geodesic and the parallel. Conversely, any (constant speed) curve satisfying ($\diamondsuit$) that is not *a parallel is a geodesic.*

Proof. For the surface of revolution parametrized as in Example 9 of Section 2, we have $E = 1$, $F = 0$, $G = f(u)^2$, $\Gamma^v_{uv} = \Gamma^v_{vu} = f'(u)/f(u)$, $\Gamma^u_{vv} = -f(u)f'(u)$, and all other Christoffel symbols are 0 (see Exercise 2.3.2d.). Then the system (♣♣) of differential equations becomes

$$(\dagger_1) \qquad u'' - ff'(v')^2 = 0$$

$$(\dagger_2) \qquad v'' + \frac{2f'}{f}u'v' = 0.$$

Rewriting the equation ($\dagger_2$) and integrating, we obtain

$$\frac{v''(t)}{v'(t)} = -\frac{2f'(u(t))u'(t)}{f(u(t))}$$
$$\ln v'(t) = -2\ln f(u(t)) + \text{const}$$
$$v'(t) = \frac{c}{f(u(t))^2},$$

so along a geodesic the quantity $f(u)^2v' = Gv'$ is constant. We recognize this as the dot product of the tangent vector of our geodesic with the vector $\mathbf{x}_v$, and so we infer that $\|\mathbf{x}_v\|\cos\phi = r\cos\phi$ is constant. (Recall that, by Proposition 4.2, the tangent vector of the geodesic has constant length.)

To this point we have seen that the equation ($\dagger_2$) is equivalent to the condition $r\cos\phi = \text{const}$, provided we assume $\|\boldsymbol{\alpha}'\|^2 = u'^2 + Gv'^2$ is constant as well. But if

$$u'(t)^2 + Gv'(t)^2 = u'(t)^2 + f(u(t))^2v'(t)^2 = \text{const},$$

we differentiate and obtain

$$u'(t)u''(t) + f(u(t))^2v'(t)v''(t) + f(u(t))f'(u(t))u'(t)v'(t)^2 = 0;$$

substituting for $v''(t)$ using ($\dagger_2$), we find

$$u'(t)\big(u''(t) - f(u(t))f'(u(t))v'(t)^2\big) = 0.$$

In other words, *provided* $u'(t) \neq 0$, a constant-speed curve satisfying ($\dagger_2$) satisfies ($\dagger_1$) as well. (See Exercise 6 for the case of the parallels.) □

Remark. We can give a simple physical interpretation of Clairaut's relation. Imagine a particle with mass 1 constrained to move along a surface. If no external forces are acting, then the particle moves along a geodesic and, moreover, angular momentum is conserved (because there are no torques). In the case of our surface of revolution, the vertical component of the angular momentum $\mathbf{L} = \boldsymbol{\alpha} \times \boldsymbol{\alpha}'$ is—surprise, surprise!—f^2v', which we've shown is constant. Perhaps some forces normal to the surface are required to keep the particle on the surface; then the particle still moves along a geodesic (why?). Moreover, since $(\boldsymbol{\alpha} \times \mathbf{n}) \cdot (0, 0, 1) = 0$, the resulting torques *still* have no vertical component.

Returning to our original motivation for geodesics, we now consider the following scenario. Choose $P \in M$ arbitrary and a geodesic $\boldsymbol{\gamma}$ through P, and draw a curve C_0 through P orthogonal to $\boldsymbol{\gamma}$. We now choose a parametrization $\mathbf{x}(u,v)$ so that $\mathbf{x}(0,0) = P$, the u-curves are geodesics orthogonal to C_0, and the v-curves are the orthogonal trajectories of the u-curves, as pictured in Figure 4.5. (It follows from Theorem

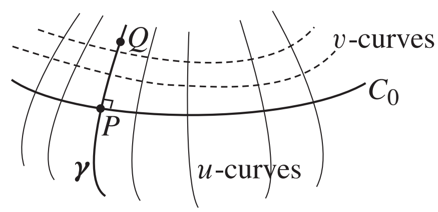


FIGURE 4.5

3.3 of the Appendix that we can do this on some neighborhood of P.)

In this parametrization we have $F = 0$ and $E = E(u)$ (see Exercise 13). Now, if $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$, $a \le t \le b$, is any path from $P = \mathbf{x}(0,0)$ to $Q = \mathbf{x}(u_0, 0)$, we have

$$\begin{aligned} \text{length}(\boldsymbol{\alpha}) &= \int_a^b \sqrt{E(u(t))u'(t)^2 + G(u(t),v(t))v'(t)^2}\,dt \ge \int_a^b \sqrt{E(u(t))}|u'(t)|\,dt \\ &\ge \int_0^{u_0} \sqrt{E(u)}\,du, \end{aligned}$$

which is the length of the geodesic arc $\boldsymbol{\gamma}$ from P to Q. Thus, we have deduced the following.

Proposition 4.5. *For any point Q on $\boldsymbol{\gamma}$ contained in this parametrization, any path from P to Q* contained in this parametrization *is at least as long as the length of the geodesic segment. More colloquially, geodesics are* locally *distance-minimizing.*

Example 7. Why is Proposition 4.5 a local statement? Well, consider a great circle on a sphere, as shown in Figure 4.6. If we go more than halfway around, we obviously have not taken the shortest path. ▽

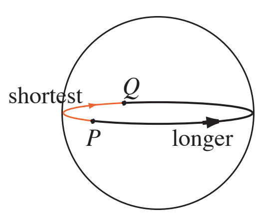


FIGURE 4.6

Remark. It turns out that any surface can be endowed with a *metric* (or *distance measure*) by defining the distance between any two points to be the infimum (usually, the minimum) of the lengths of all piecewise-$\mathcal{C}^1$ paths joining them. (Although the distance measure is different from the Euclidean distance as the surface sits in $\mathbb{R}^3$, the topology—notion of "neighborhood"—induced by this metric structure is the induced topology that the surface inherits as a subspace of $\mathbb{R}^3$.) It is a consequence of the Hopf-Rinow Theorem (see M. doCarmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976, p. 333, or M. Spivak, *A*

Comprehensive Introduction to Differential Geometry, third edition, volume 1, Publish or Perish, Inc., 1999, p. 342) that in a surface in which every parametrized geodesic is defined for all time (a "complete" surface), every two points are in fact joined by a geodesic of least length. The proof of this result is quite tantalizing: To find the shortest path from P to Q, one walks around the "geodesic circle" of points a small distance from P and finds the point R on it closest to Q; one then proves that the unique geodesic emanating from P that passes through R must eventually pass through Q, and there can be no shorter path.

We referred earlier to two surfaces M and M^* as being globally isometric (e.g., in Example 6 in Section 1). We can now give the official definition: There should be a function $f\colon M \to M^*$ that establishes a one-to-one correspondence and preserves distance—for any $P, Q \in M$, the distance between P and Q in M should be equal to the distance between $f(P)$ and $f(Q)$ in M^*.

EXERCISES 2.4

1. Determine the result of parallel translating the vector $(0, 0, 1)$ once around the circle $x^2 + y^2 = a^2$, $z = 0$, on the right circular cylinder $x^2 + y^2 = a^2$.

2. Prove that $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

3. Suppose $\boldsymbol{\alpha}$ is a non-arclength-parametrized curve. Using the formula $(**)$ on p. 14, prove that the velocity vector of $\boldsymbol{\alpha}$ is parallel along $\boldsymbol{\alpha}$ if and only if $\kappa_g = 0$ and $\upsilon' = 0$.

*4. Find the geodesic curvature κ_g of a latitude circle $u = u_0$ on the unit sphere (see Example 1(d) on p. 37)
   a. directly
   b. by applying the result of Exercise 2

5. Consider the right circular cone with vertex angle 2ϕ parametrized by

$$\mathbf{x}(u, v) = (u \tan\phi \cos v, u \tan\phi \sin v, u), \quad 0 < u \le u_0,\ 0 \le v \le 2\pi.$$

   Find the geodesic curvature κ_g of the circle $u = u_0$ by using trigonometric considerations. Check that your answer agrees with the curvature of the circle you get by unrolling the cone to form a "pacman" figure, as shown on the left in Figure 4.7. (For a proof that these curvatures should agree, see Exercise 2.1.10 and Exercise 3.1.7.)

6. Check that the parallel $u = u_0$ is a geodesic on the surface of revolution parametrized as in Proposition 4.4 if and only if $f'(u_0) = 0$. Give a geometric interpretation of and explanation for this result.

7. Use the equations (♣), as in Example 3, to determine through what angle a vector turns when it is parallel-translated once around the circle $u = u_0$ on the cone $\mathbf{x}(u, v) = (u \cos v, u \sin v, cu)$, $c \neq 0$. (See Exercise 2.3.2c.)

8. a. Prove that if the surfaces M and M^* are tangent along the curve C, parallel translation along C is the same in both surfaces.