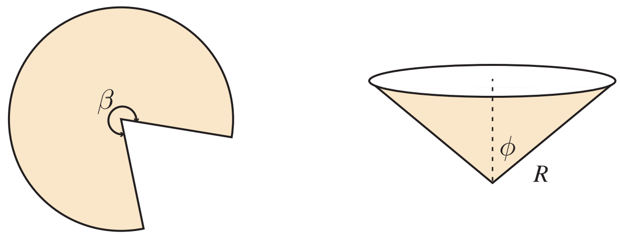

FIGURE 1.11

c. Show that we obtain the same result by "smoothing" the cone point, as pictured in Figure 1.12. (Hint: Interpret $\iint_R K\,dA$ as the area of the image of the Gauss map.)

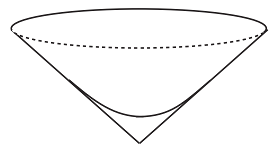

FIGURE 1.12

Remark. It is not hard to give an explicit $\mathcal{C}^2$ such smoothing. For example, construct a $\mathcal{C}^2$ convex function f on $[0,1]$ with $f(0) = f'(0) = 0$, $f(1) = f'(1) = 1$, and $f''(1) = 0$.

14. Suppose $\boldsymbol{\alpha}$ is a closed space curve with $\kappa \neq 0$. Assume that the *normal indicatrix* (i.e., the curve traced out on the unit sphere by the principal normal) is a simple closed curve in the unit sphere. Prove then that it divides the unit sphere into two regions of equal area. (Hint: Apply the Gauss-Bonnet Theorem to one of those regions.)

15. Suppose $M \subset \mathbb{R}^3$ is a compact, oriented surface with no boundary with $K > 0$. It follows that M is topologically a sphere (why?). Prove that M is convex; i.e., for each $P \in M$, M lies on only one side of the tangent plane $T_P M$. (Hint: Use the Gauss-Bonnet Theorem and Gauss's original interpretation of curvature indicated in the remark on p. 51 to show the Gauss map must be one-to-one (except perhaps on a subset with no area). Then look at the end of the proof of Theorem 3.4 of Chapter 1.)

2. An Introduction to Hyperbolic Geometry

Hilbert proved in 1901 that there is no surface (without boundary) in $\mathbb{R}^3$ with constant negative curvature with the property that it is a closed subset of $\mathbb{R}^3$ (i.e., every Cauchy sequence of points in the surface converges to a point of the surface). The pseudosphere fails the latter condition. Nevertheless, it is possible to give a definition of an "abstract surface" (not sitting inside $\mathbb{R}^3$) together with a first fundamental form. As we know, this will be all we need to calculate Christoffel symbols, curvature (Theorem 3.1 of Chapter 2), geodesics, and so on.

Definition. The *hyperbolic plane* $\mathbb{H}$ is defined to be the half-plane $\{(u, v) \in \mathbb{R}^2 : v > 0\}$, equipped with the first fundamental form I given by $E = G = 1/v^2$, $F = 0$.

Now, using the formulas (‡) on p. 58, we find that

$$\begin{aligned} \Gamma^u_{uu} &= \frac{E_u}{2E} = 0 & \Gamma^v_{uu} &= -\frac{E_v}{2G} = \frac{1}{v} \\ \Gamma^u_{uv} &= \frac{E_v}{2E} = -\frac{1}{v} & \Gamma^v_{uv} &= \frac{G_u}{2G} = 0 \\ \Gamma^u_{vv} &= -\frac{G_u}{2E} = 0 & \Gamma^v_{vv} &= \frac{G_v}{2G} = -\frac{1}{v}. \end{aligned}$$

Using the formula (∗) for Gaussian curvature on p. 60, we find

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) = -\frac{v^2}{2}\left(-\frac{2}{v^3}\cdot v^2\right)_v = -\frac{v^2}{2}\cdot\frac{2}{v^2} = -1.$$

Thus, the hyperbolic plane has constant curvature -1. Note that it is a consequence of Corollary 1.7 that the area of a geodesic triangle in $\mathbb{H}$ is equal to $\pi - (\iota_1 + \iota_2 + \iota_3)$.

What are the geodesics in this surface? Using the equations (♣♣) on p. 71, we obtain the equations

$$u'' - \frac{2}{v}u'v' = v'' + \frac{1}{v}(u'^2 - v'^2) = 0.$$

Obviously, the vertical rays $u = \text{const}$ give us solutions (with $v(t) = c_1e^{c_2t}$). Next we seek geodesics with $u' \neq 0$, so we start with $\dfrac{dv}{du} = \dfrac{v'}{u'}$ and apply the chain rule judiciously:

$$\begin{aligned} \frac{d^2v}{du^2} &= \frac{d}{du}\left(\frac{v'}{u'}\right) = \frac{u'v'' - u''v'}{u'^2}\cdot\frac{1}{u'} \\ &= \frac{1}{u'^3}\left(u'\left(\frac{1}{v}\right)(v'^2 - u'^2) - v'\left(\frac{2}{v}u'v'\right)\right) \\ &= -\frac{1}{v}\left(1 + \left(\frac{v'}{u'}\right)^2\right) = -\frac{1}{v}\left(1 + \left(\frac{dv}{du}\right)^2\right). \end{aligned}$$

This means we are left with the differential equation

$$v\frac{d^2v}{du^2} + \left(\frac{dv}{du}\right)^2 = \frac{d}{du}\left(v\frac{dv}{du}\right) = -1,$$

and integrating this twice gives us the solutions

$$u^2 + v^2 = au + b.$$

That is, the geodesics in $\mathbb{H}$ are the vertical rays and the semicircles centered on the u-axis, as pictured in Figure 2.1. Note that any semicircle centered on the u-axis intersects each vertical line at most one time. It now follows that any two points $P, Q \in \mathbb{H}$ are joined by a unique geodesic. If P and Q lie on a vertical line, then the vertical ray through them is the unique geodesic joining them. If P and Q do not lie on a vertical line, let C be the intersection of the perpendicular bisector of $\overline{PQ}$ and the u-axis; then the semicircle centered at C is the unique geodesic joining P and Q.

Example 1. Given $P, Q \in \mathbb{H}$, we would like to find a formula for the (geodesic) distance $d(P, Q)$ between them. Let's start with $P = (u_0, a)$ and $Q = (u_0, b)$, with $0 < a < b$. Parametrizing the line segment from P to Q by $u = u_0, v = t, a \le t \le b$, we have

$$d(P, Q) = \int_a^b \sqrt{Eu'(t)^2 + Gv'(t)^2}\,dt = \int_a^b \frac{dt}{t} = \ln\frac{b}{a}.$$

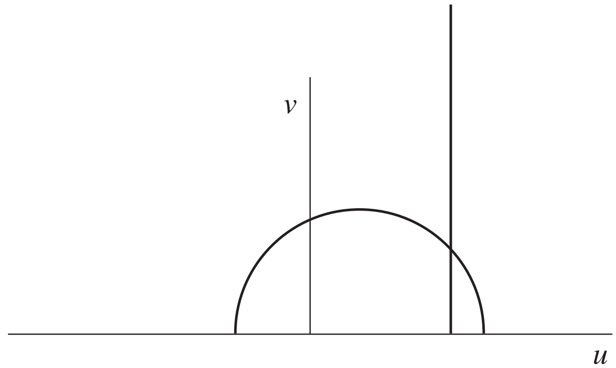

FIGURE 2.1

Note that, fixing Q and letting P approach the u-axis, $d(P,Q) \to \infty$; thus, it is reasonable to think of points on the u-axis as "virtual" points at infinity.

In general, we parametrize the arc of a semicircle $(u_0 + r\cos t, r\sin t)$, $\theta_1 \le t \le \theta_2$, going from P to

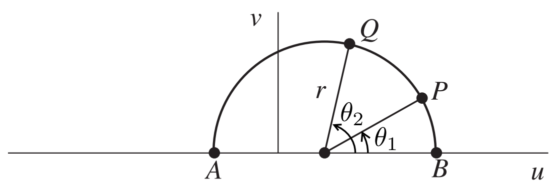

FIGURE 2.2

Q, as shown in Figure 2.2. Then we have

$$\begin{aligned} d(P,Q) &= \left|\int_{\theta_1}^{\theta_2} \sqrt{Eu'(t)^2 + Gv'(t)^2}dt\right| = \left|\int_{\theta_1}^{\theta_2} \frac{r\,dt}{r\sin t}\right| = \left|\int_{\theta_1}^{\theta_2} \frac{dt}{\sin t}\right| \\ &= \left|\ln\left(\frac{1+\cos\theta_1}{\sin\theta_1}\Big/\frac{1+\cos\theta_2}{\sin\theta_2}\right)\right| = \left|\ln\left(\frac{2\cos(\theta_1/2)}{2\sin(\theta_1/2)}\Big/\frac{2\cos(\theta_2/2)}{2\sin(\theta_2/2)}\right)\right| \\ &= \left|\ln\left(\frac{AP}{BP}\Big/\frac{AQ}{BQ}\right)\right|, \end{aligned}$$

where the lengths in the final formula are Euclidean. (See Exercise 12 for the connection with cross ratio.) ∇

It follows from the first part of Example 1 that the curves $v = a$ and $v = b$ are a constant distance apart (measured along geodesics orthogonal to both), like parallel lines in Euclidean geometry. These curves are classically called *horocycles*. As we see in Figure 2.3, these curves are the curves orthogonal to the family of the "vertical geodesics." If, instead, we consider all the geodesics passing through a given point Q "at infinity" on $v = 0$, as we ask the reader to check in Exercise 5, the orthogonal trajectories will be curves in $\mathbb{H}$ represented by circles tangent to the u-axis at Q.

Example 2. Let's calculate the geodesic curvature of the horocycle $v = a$, oriented to the right. We start by parametrizing the curve by $\boldsymbol{\alpha}(t) = (t, a)$. Then $\boldsymbol{\alpha}'(t) = (1, 0)$. Note that $\upsilon(t) = \|\boldsymbol{\alpha}'(t)\| =$

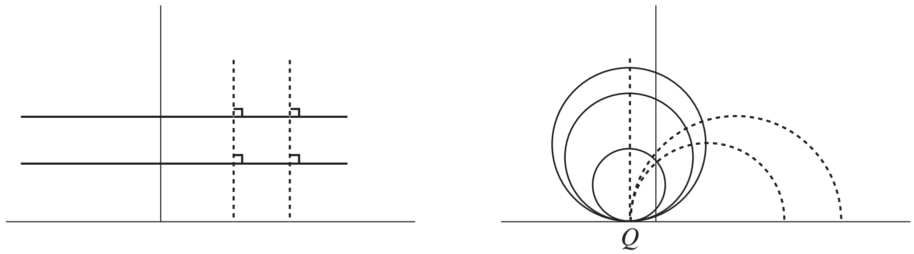

FIGURE 2.3

$\sqrt{E(1)^2 + G(0)^2} = 1/a$. By Proposition 1.1,

$$\phi_{12} = \frac{1}{2\sqrt{\frac{1}{a^4}}}(2a^{-3} \cdot 1) = \frac{1}{a}.$$

(Here $\mathbf{e}_1 = v(1,0)$ and $\mathbf{e}_2 = v(0,1)$ at the point $(u,v) \in \mathbb{H}$. Why?) To calculate the geodesic curvature, we wish to apply Proposition 1.3, which requires differentiation with respect to arclength, so we'll use the chain rule as in Chapter 1, multiplying the t-derivative by $1/v(t) = a$. Note, also, that $\boldsymbol{\alpha}'$ makes the constant angle $\theta = 0$ with $\mathbf{e}_1$, so $\theta' = 0$. Thus,

$$\kappa_g = \frac{1}{v(t)}\phi_{12} = a \cdot \frac{1}{a} = 1,$$

as required. (Note that if we move to the left, the sign changes and $\kappa_g = -1$.) ∇

We ask the reader to do the analogous calculations for the circles tangent to the u-axis in Exercise 6. Moreover, as we ask the reader to check in Exercise 7, every curve in $\mathbb{H}$ of constant geodesic curvature $\kappa_g = \pm 1$ is a horocycle.

Remark. It seems somewhat surprising to find in Example 2 that $\phi_{12} = 1/a$, as $\mathbf{e}_1$ certainly doesn't appear to be turning as we move along the path. However, as we discussed in the Remark on p. 71, at any point of $v = a$ the geodesic with the same tangent vector is a semicircle heading "to the right," and so this means that $\mathbf{e}_1$ is turning to the left, i.e., towards $\mathbf{e}_2$.

The isometries of the Euclidean plane form a group, the Euclidean group $E(2)$; the isometries of the sphere likewise form a group, the orthogonal group $O(3)$. Each of these is a 3-dimensional Lie group. Intuitively, there are three degrees of freedom because we must specify where a point P goes (two degrees of freedom) and where a single unit tangent vector at that point P goes (one more degree of freedom). We might likewise expect the isometries of $\mathbb{H}$ to form a 3-dimensional group. And indeed it is. We deal with just the orientation-preserving isometries here.

We consider $\mathbb{H} \subset \mathbb{C}$ by letting (u,v) correspond to $z = u + iv$, and we consider the collection of *linear fractional transformations*

$$T(z) = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R}, \quad ad - bc = 1.$$

We must now check several things:

(i) Composition of functions corresponds to multiplication of the 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with determinant 1, so we obtain a group.

(ii) T maps $\mathbb{H}$ bijectively to $\mathbb{H}$.
(iii) T is an isometry of $\mathbb{H}$.

We leave it to the reader to check the first two in Exercise 8, and we check the third here. Given the point $z = u + iv$, we want to compute the lengths of the vectors T_u and T_v at the image point $T(z) = x + iy$ and see that the two vectors are orthogonal. Note that

$$\frac{az+b}{cz+d} = \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{\big(a(u+iv)+b\big)\big(c(u-iv)+d\big)}{|cz+d|^2}$$
$$= \frac{\big(ac(u^2+v^2)+(ad+bc)u+bd\big)+i\big((ad-bc)v\big)}{|cz+d|^2},$$

so $y = \dfrac{v}{|cz+d|^2}$. Now we have[3]

$$x_u + iy_u = -ix_v + y_v = T'(z) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{1}{(cz+d)^2},$$

so we have

$$\tilde{E} = \frac{x_u^2+y_u^2}{y^2} = \frac{1}{y^2}|T'(z)|^2 = \frac{1}{y^2}\cdot\frac{1}{|cz+d|^4} = \frac{1}{v^2} = E,$$

and, similarly, $\tilde{G} = \dfrac{x_v^2+y_v^2}{y^2} = G$. On the other hand,

$$\tilde{F} = \frac{x_u y_u + x_v y_v}{y^2} = \frac{x_u(-x_v)+x_v(x_u)}{y^2} = 0 = F,$$

as desired.

Now, as we verify in Exercise 12 or in Exercise 14, linear fractional transformations carry lines and circles in $\mathbb{C}$ to either lines or circles. Since our particular linear fractional transformations preserve the real axis ($\cup\{\infty\}$) and preserve angles as well, it follows that vertical lines and semicircles centered on the real axis map to one another. Thus, our isometries do in fact map geodesics to geodesics (how comforting!).

If we think of $\mathbb{H}$ as modeling non-Euclidean geometry, with lines in our geometry being the geodesics, note that given any line ℓ and point $P \notin \ell$, there are *infinitely many* lines passing through P "parallel" to (i.e., not intersecting) ℓ. As we see in Figure 2.4, there are two special lines through P that "meet ℓ at

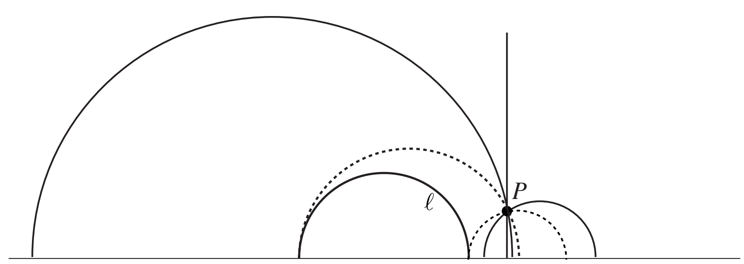

FIGURE 2.4

infinity"; the rest are often called *ultraparallels*.

We conclude with an interesting application. As we saw in the previous section, the Gauss-Bonnet Theorem gives a deep relation between the total curvature of a surface and its topological structure (Euler

[3]These are the Cauchy-Riemann equations from basic complex analysis.

characteristic). We know that if a compact surface M is topologically equivalent to a sphere, then its total curvature must be that of a round sphere, namely 4π. If M is topologically equivalent to a torus, then (as the reader checked in Exercise 3.1.3) its total curvature must be 0. We know that there is no way of making

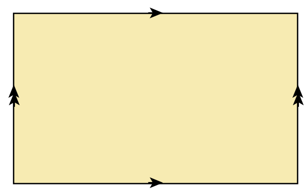

FIGURE 2.5

the torus in $\mathbb{R}^3$ in such a way that it has constant Gaussian curvature $K = 0$ (why?), but we *can* construct a flat torus in $\mathbb{R}^4$ by taking

$$\mathbf{x}(u, v) = (\cos u, \sin u, \cos v, \sin v), \qquad 0 \le u, v \le 2\pi.$$

(We take a piece of paper and identify opposite edges, as indicated in Figure 2.5; this can be rolled into a cylinder in $\mathbb{R}^3$ but into a torus only in $\mathbb{R}^4$.) So what happens with a 2-holed torus? In that case, $\chi(M) = -2$, so the total curvature should be -4π, and we can reasonably ask if there's a 2-holed torus with *constant* negative curvature. Note that we can obtain a 2-holed torus by identifying pairs of edges on an octagon, as

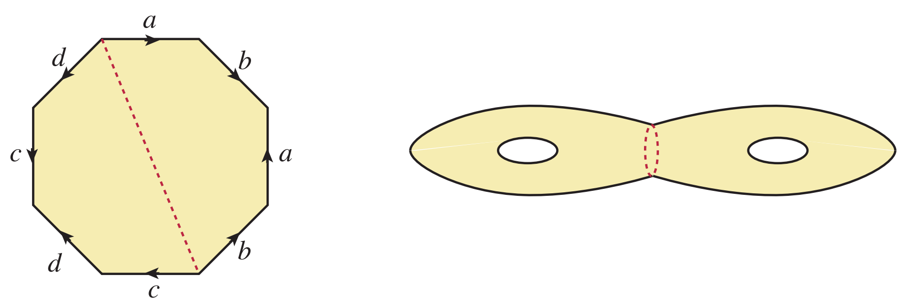

FIGURE 2.6

shown in Figure 2.6.

This leads us to wonder whether we might have regular n-gons R in $\mathbb{H}$. By the Gauss-Bonnet formula, we would have $\text{area}(R) = (n-2)\pi - \sum \iota_j$, so it's obviously necessary that $\sum \iota_j < (n-2)\pi$. This shouldn't be difficult so long as $n \ge 3$. First, let's convince ourselves that, given any point $P \in \mathbb{H}$, $0 < \alpha < \pi$, and $0 < \beta < (\pi - \alpha)/2$, we can construct an isosceles triangle with vertex angle α at P and base angle β. We draw two geodesics emanating from P with angle α between them, as shown in Figure 2.7. Proceeding a geodesic distance r on each of them to points Q and R, we then obtain an isosceles triangle $\triangle PQR$ with vertex angle α. Now, the base angle of that triangle approaches $(\pi - \alpha)/2$ as $r \to 0^+$ and approaches 0 as $r \to \infty$. It follows (presuming that the angle varies continuously with r) that for some r, we obtain the desired base angle β. Let's now apply this construction with $\alpha = 2\pi/n$ and $\beta = \pi/n$, $n \ge 5$. Repeating the construction n times (dividing the angle at P into n angles of $2\pi/n$ each), we obtain a regular n-gon with the property that $\sum \iota_j = 2\pi$, as shown (approximately?) in Figure 2.8 for the case $n = 8$. The point is that because the interior angles add up to 2π, when we identify edges as in Figure 2.6, we will obtain a

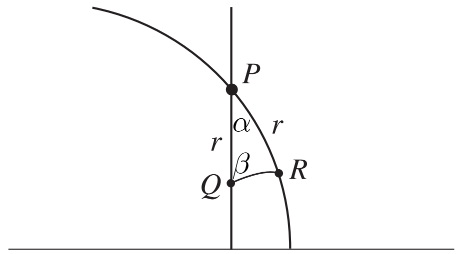

FIGURE 2.7

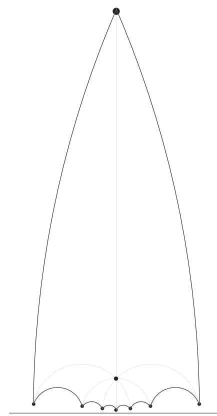

FIGURE 2.8

smooth 2-holed torus with constant curvature $K = -1$. The analogous construction works for the g-holed torus, constructing a regular $4g$-gon whose interior angles sum to 2π.

EXERCISES 3.2

1. Find the geodesic joining P and Q in $\mathbb{H}$ and calculate $d(P, Q)$.
   a. $P = (4, 3)$, $Q = (-3, 4)$
   *b. $P = (1, 2)$, $Q = (0, 1)$
   c. $P = (20, 7)$, $Q = (16, 15)$
2. Suppose there is a geodesic perpendicular to two geodesics in $\mathbb{H}$. What can you prove about the latter two?
3. Prove the angle-angle-angle *congruence* theorem for hyperbolic (geodesic) triangles: If $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$, then $\triangle ABC \cong \triangle A'B'C'$. (Hint: Use an isometry to move A' to A, B' along the geodesic from A to B, and C' along the geodesic from A to C.)
4. a. Verify Local Gauss-Bonnet, Theorem 1.6, for the region R bounded by $u = A$, $u = B$, $v = a$, and $v = b$.
   b. Verify Local Gauss-Bonnet for the region R bounded by the segment $v = a$, $A \le u \le B$, and the geodesic joining the two endpoints.