4. The Inverse Function Theorem

Suppose we have a smooth map

$$f: X \to Y$$

where X and Y are manifolds of the same dimension. (If $\exists$ an open $U \ni x$ and $V \ni f(x) = y$ so that

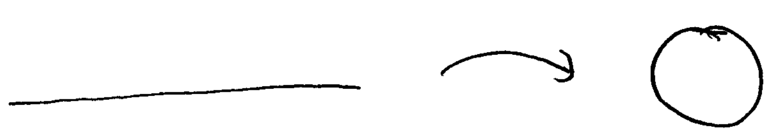
$$f: U \to V \text{ is a diffeomorphism})$$

we say f is a <u>local diffeomorphism</u> at x.

> **Inverse Function Theorem.** Suppose $f: X \to Y$ is a smooth map. Then
>
> df_x is an isomorphism $\iff$ f is a local diffeomorphism at x

Things to notice:

f can be a local diffeomorphism at every point without being a diffeomorphism



(Not that being a loc. diffeo. has no consequences... such a map is called a covering map.)

If we choose the right local coordinates,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\text{identity}"] & U \arrow[u, "\psi"'] \end{tikzcd}$$

we have $f(\vec{x}) = \vec{x}$ near 0.

Definition. $f: X \to Y$ and $f': X' \to Y'$ are <u>equivalent</u> maps if $\exists$ diffeomorphisms α and β so that

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[u, "\alpha"'] \arrow[r, "f'"] & Y' \arrow[u, "\beta"'] \end{tikzcd}$$

commutes.

so IFT states:

df_x is an isomorphism $\iff$ f is locally equivalent to Id

What if $\dim X \neq \dim Y$? Suppose $\dim Y > \dim X$.

Definition. $f : X \to Y$ is an <u>immersion</u> ~~if~~ at x if $df_x : T_x X \to T_{f(x)} Y$ is <u>injective</u>.

We have

Local Immersion Theorem. Suppose $f : X \to Y$ is an immersion at x, and $y = f(x)$. There are local coordinates around x and y so that

$$f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0).$$

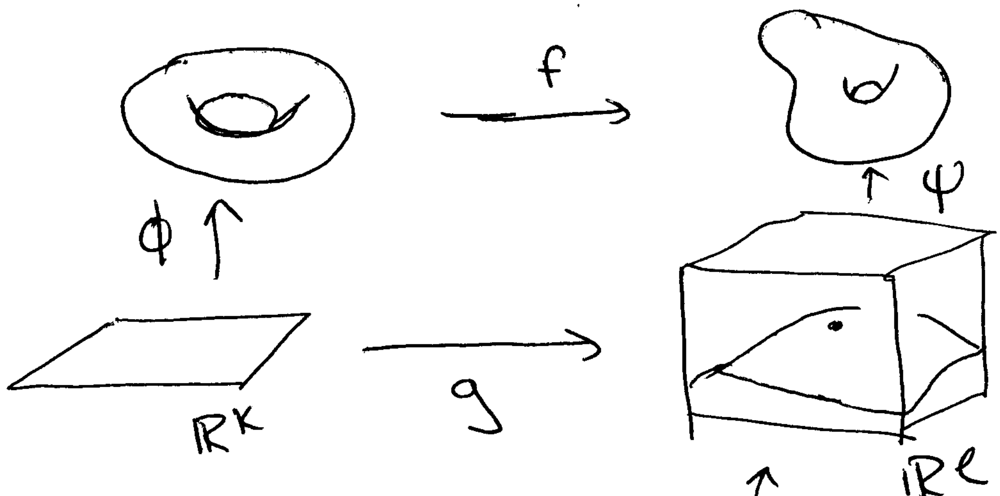
Proof. Choose local coords

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \subset \mathbb{R}^k \arrow[u, "\varphi"] \arrow[r, "g"'] & V \subset \mathbb{R}^l \arrow[u, "\psi"'] \end{tikzcd}$$

Now choose coordinates ~~in~~ on $V \subset \mathbb{R}^\ell$ so that

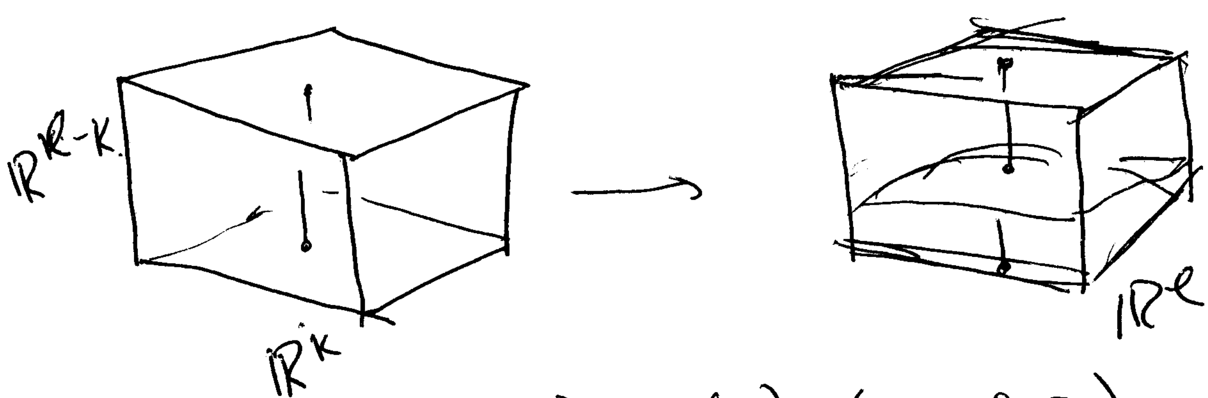
$$dg_0 : \mathbb{R}^k \to \mathbb{R}^\ell$$

~~maps~~ has a matrix in the form $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$.



orient $\mathbb{R}^\ell$ so that tangent plane to $g(\mathbb{R}^k)$ at $g(0)=0$ is $\mathrm{span}(e_1, \dots, e_k)$.

Now define $G : \mathbb{R}^k \times \mathbb{R}^{k-\ell} \to \mathbb{R}^\ell$

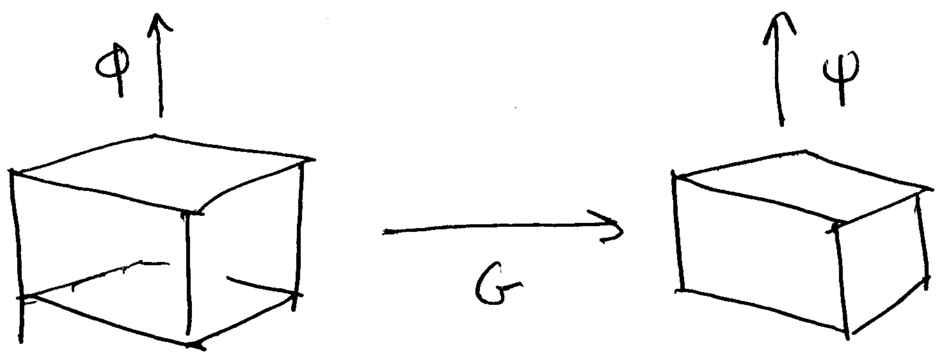


$$G(x,z) = g(x) + (0, \dots, 0, z)$$

at 0,

$$dG = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & I_{\ell-k} \end{array}\right) = I$$

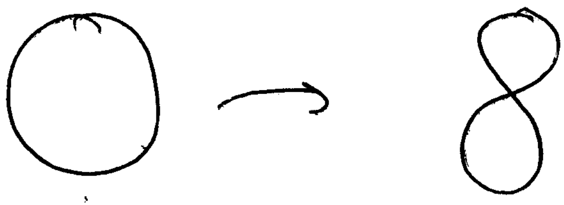
so G is a local diffeomorphism by IFT.
This means that $G \circ \varphi$ is a local diffeomorphism.

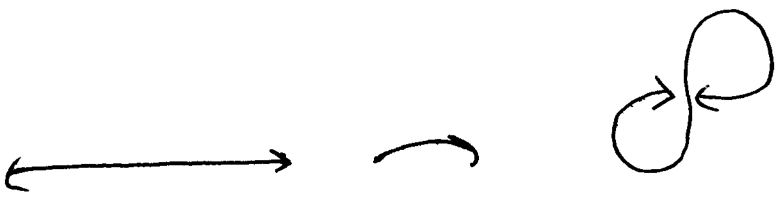


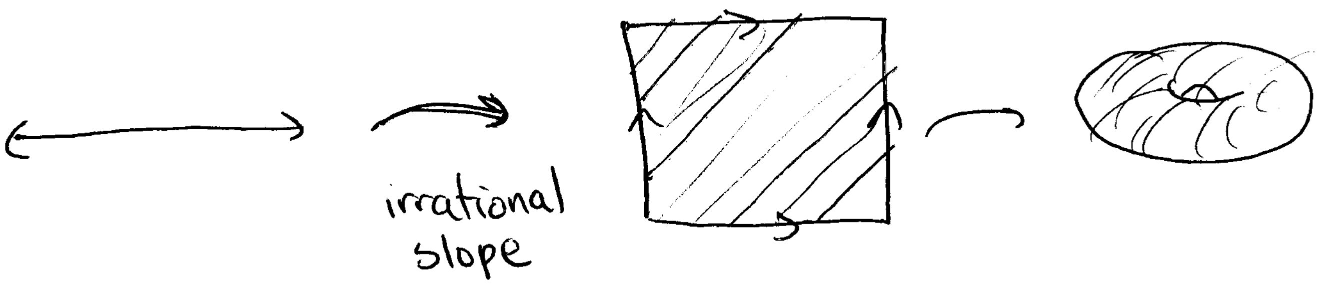
But these are the coordinates we wanted on Y to make f look like we wanted!

Q: what does the image of an immersion look like?

A: Could be pretty weird.

a) 

b) 

c) 

How can we control this pheomenon?

Definition: A map $f: X \to Y$ is <u>proper</u> if the preimage of every compact set in Y is compact in X.

Definition. If $f: X \to Y$ is ~~an~~ injective, proper, & an immersion, then f is an <u>embedding</u>.

Theorem. An embedding $f: X \to Y$ maps X diffeomorphically to a submanifold $f(X)$ of Y.

Proof. First, we observe that if we knew $f(W)$ open in $f(X)$ for each open $W \subset X$, we'd have shown that $f(X)$ is a manifold.

~~By the~~

Choose $y \in f(X)$. Since f is 1-1, $\exists$ x s.t. $y = f(x)$. By the local immersion theorem, $\exists$ a neighborhood W of x so that W and $f(W)$ are diffeomorphic.

If $f(W)$ was a neighborhood of $f(x)$ in $f(X)$, we'd have just shown $f(X)$ is a manifold.

Suppose not. Then $\exists$ $y_i \in f(X)$ s.t. $y_i \to y$ with $y_i \notin f(W)$ and $y \in f(W)$. Consider the set $\{y_i, y\}$, which is clearly compact.

Since f is proper, and 1-1, $\{y_i, y\} = f\{x_i, x\}$ where $\{x_i, x\}$ is compact.

Extract a convergent subsequence of x_i, converging to some z. Then

$$f(x_i) \to f(z) \quad \text{since } \cancel{f}\ x_i \to z.$$

But $f(x_i) \to f(x)$, so $x = z$ and $z \in W$. But W is <u>open</u>, so for large i we must have $x_i \in W$. This contradicts our assumption that $f(x_i) = y_i \notin f(W)$. $\times\times$

So $f(X)$ is a manifold. But f is 1-1 and onto $f(X)$, so f^{-1} is defined on $f(X)$. By local immersion theorem, we know f^{-1} is smooth! $\therefore$

Hwk: Section 3

3, 9

Next week:

§4 1, 2, 9, 10

§5 2, 4, 10

§6 2, 7, 10