Transversality Theorem.

Suppose $F: X \times S \to Y$ is a smooth map of manifolds where only X has boundary and let $Z \subset Y$ be a submanifold without boundary.

If $F, \partial F$ transversal to Z, then for almost any $s \in S$, $f_s = F(-, s)$ and ∂f_s are transversal to Z.

~~"A small homotopy of any m~~

Proof. We know

$W = F^{-1}(Z)$ is a mfld with boundary $W \cap \partial(X \times S)$.

Consider

$$\pi: X \times S \to S, \quad \pi(x,s) = s.$$

Claim. If s is a regular value of $\pi|_W$ and of $\partial\pi|_{\partial W}$, then $f_s \pitchfork Z$.

Take (x,s) so that $f_s(x) = z \in Z$. We must show that

$$d(f_s)_x \, T_{(x,s)}(X \times \{s\}) + T_z Z = T_z Y.$$

or, for any $\vec{a} \in T_z Y$, we must show $\exists \; \vec{v} \in d(f_s)_x \, T_{(x,s)}(X \times \{s\})$ so that

$$\vec{a} - \underbrace{\cancel{\vec{v}}}_{d(f_s)_x(\vec{v})} \in T_z Z.$$

We know that for this $\vec{a}$, $\exists \; \vec{b} \in T_{(x,s)} X \times S$ so that

$$\vec{a} - dF_{(x,s)}(\vec{b}) \in T_z Z$$

(transversality of F w.r.t. Z). Now

$$T_{(x,s)} X \times S = T_x X \times T_s S$$

so our

$$\vec{b} = (\vec{\omega}, \vec{e}) \quad \text{for } \vec{\omega} \in T_x X, \ \vec{e} \in T_s S.$$

If $\vec{e} = 0$, then $\vec{\omega}$ would be our $\vec{v}$ since

$$dF_{(x,s)}(\vec{\omega}, 0) = df_s(\vec{\omega}).$$

Plan: Modify $\vec{b}$ to cancel $\vec{e}$.

We know that

$$d\pi_{(x,s)} : \underset{T_x X \times T_s S}{\underset{\shortparallel}{T_{(x,s)}(X \times S)}} \longrightarrow T_s S$$

is just the projection $(\vec{\omega}, \vec{e}) \to \vec{e}$. But $d\pi$ maps $T_{(x,s)} W$ onto $T_s S$ since s is a regular value of $\pi|_W$.

Thus $\exists$ $(\vec{u}, \vec{e})$ so that $(\vec{u}, \vec{e}) \in T_{(x,s)} W$.

Observe that $F(\omega) \subset Z$, so

$$dF_{(x,s)}(\vec{v}, \vec{e}) \in T_z Z.$$

We claim

$$\vec{V} = \vec{\omega} - \vec{u}$$

has the property that

$$a - df_s(\vec{v}) \in T_z Z.$$

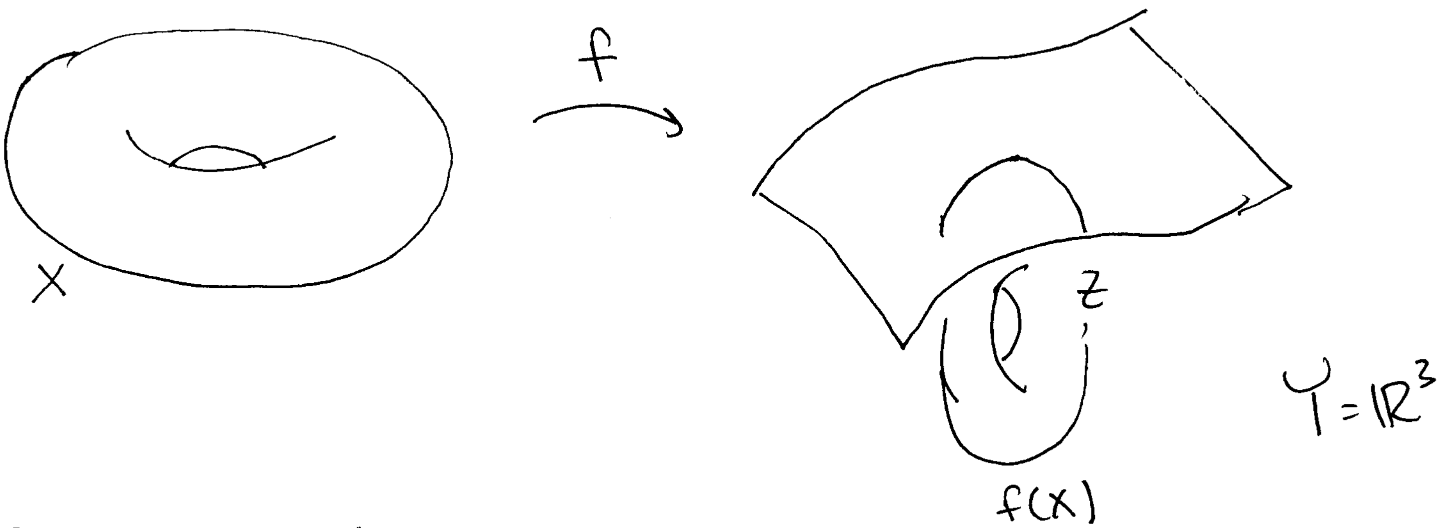
We check

$$\begin{aligned}
a - df_s(\vec{v}) &= a - dF_{(x,s)}(\vec{v}, 0) \\
&= a - dF_{(x,s)}(\vec{\omega} - \vec{u}, \vec{e} - \vec{e}) \\
&= a - \left[dF_{(x,s)}(\vec{\omega}, \vec{e})\right] + dF_{(x,s)}(\vec{v}, \vec{e}) \\
&= \underbrace{a - dF_{(x,s)}(\vec{b})}_{\text{in } T_z Z \text{ by construction of } \vec{b}} + \underbrace{dF_{(x,s)}(\vec{v}, \vec{e})}_{\text{in } T_z Z \text{ by construction of } (\vec{v}, \vec{e})}
\end{aligned}$$

$$\in T_z Z.$$

Since Sard's theorem says almost every $s \in S$ is a regular value, we're done.

Now imagine



If we build

$$F : X \times \mathbb{R}^3 \to Y \quad \text{by} \quad F(x, \vec{s}) = f(x) + \vec{s}$$

then F is a submersion onto Y so F is <u>always</u> transverse to any $Z \subset Y$.

By the Theorem above, almost every variation $\vec{s} \in \mathbb{R}^3$ makes the map $f(x)+s$ transverse to Z.

Idea: Transverse maps are the <u>only</u> observable maps.

What if Y is <u>not</u> Euclidean space? We need to pretend it is...

ϵ-Neighborhood theorem.