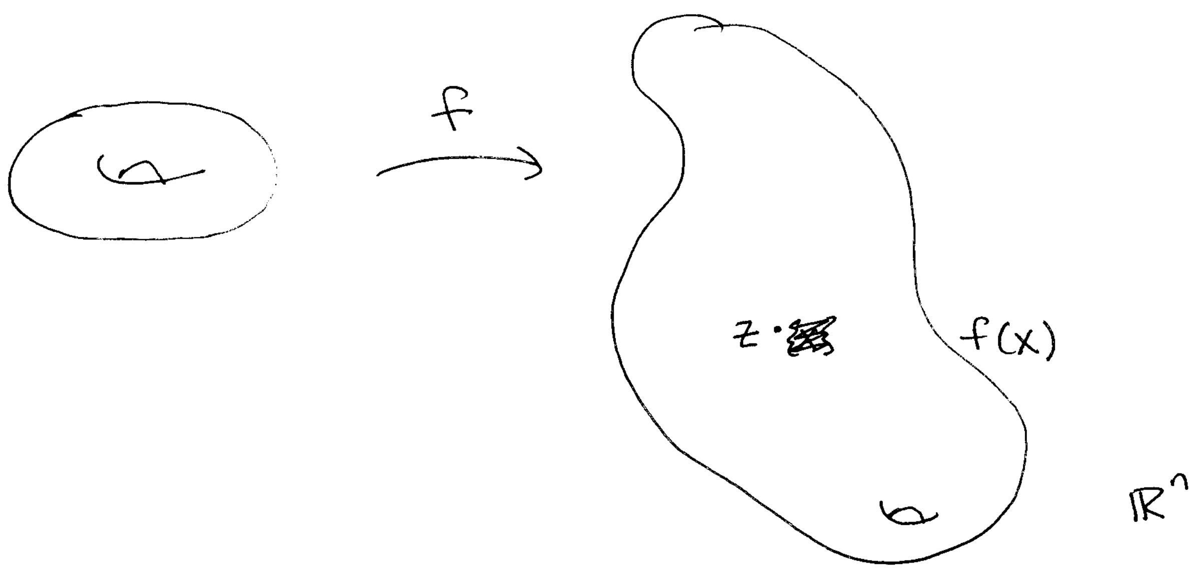
Winding Numbers and more!

Suppose we have some compact, connected $n-1$ manifold ~~contained~~ in X and a smooth map $f: X \to \mathbb{R}^n$.



We want to consider how $f(X)$ wraps in $\mathbb{R}^n$, so pick some $z \notin f(X)$ and consider

$$u(x) = \frac{f(x) - z}{|f(x) - z|} : X \to S^{n-1}$$

(this is the trick we used in the Fundamental Theorem of algebra proof last time.).

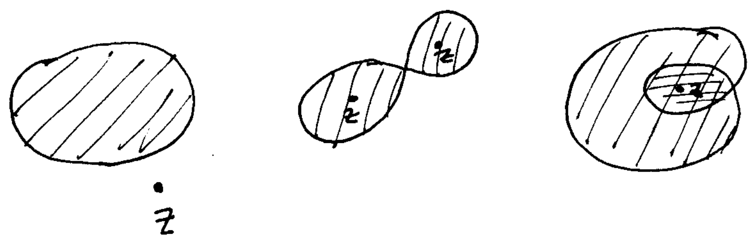
We know

$\deg_2 u$ is a homotopy invariant of u.

Definition. $\deg_2 u$ is called the <u>mod 2 winding number of f around z</u>, and this is written

$$W_2(f, z).$$

We first claim



Theorem. Suppose $X = \partial D$ and let $F : D \to \mathbb{R}^n$ extend $f : X \to \mathbb{R}^n$. If z is a regular value of F and $z \notin f(X)$, then

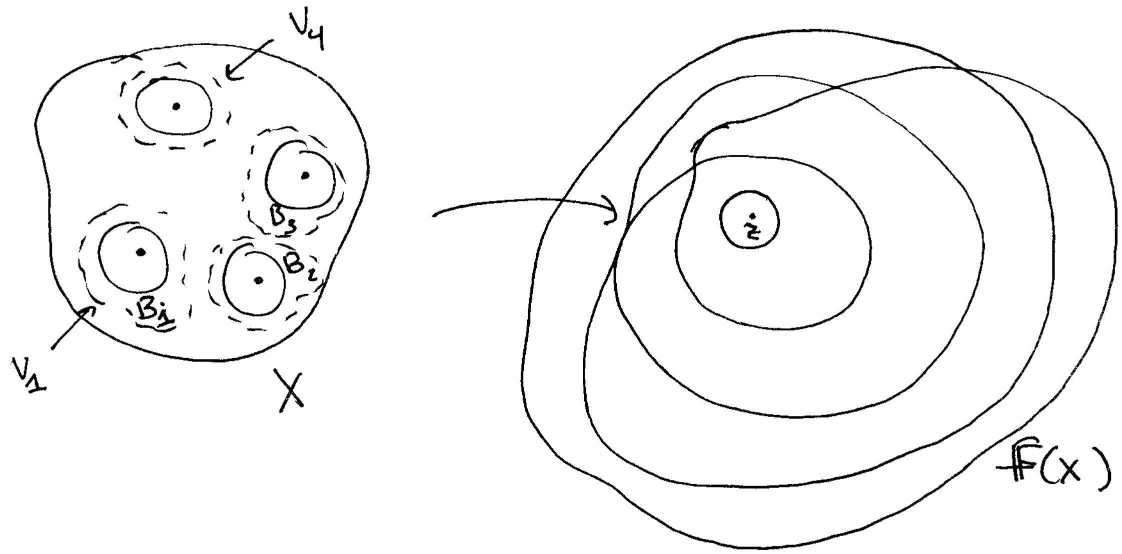
$$W_2(f, z) = \#\text{ of points in } F^{-1}(z).$$

Proof. We observe that

$$W_2(f, z) = \deg_2 u = I_2(u, \{v\})$$

for a direction $v \in S^{n-1}$. But if u extends to D, then $I_2(u, \{v\}) = 0$ by our Boundary Theorem from last class.

So if $z \notin F(D)$, we're done. Suppose $z \in F(D)$. By stack of records, there are open sets in D called $V_1, \dots, V_n$ mapping diffeomorphically to an open ~~contai~~ set U containing z.



Now if we take little balls B_i around each of the points in $F^{-1}(z)$, then ~~u does~~ extend ~~to $D-(B_1 \sqcup \dots \sqcup B_n)$ so we can def~~

~~$\mathbb{Z}_2(u$~~

we can define $u_i : \partial B_i \to S^{n-1}$ by

$$u_i(x) = \frac{F(x)-z}{|F(x)-z|}$$

and the collection of maps $u, u_1, \dots, u_n$ extend to $D-(B_1 \sqcup \dots \sqcup B_n)$.

But this means that for $v \in S_1$,

$$I_2(u,\{v\}) + I_2(u_1,\{v\}) + \ldots + I_2(u_n,\{v\}) = 0 \mod 2$$

or if $f_i = F|_{\partial B}$

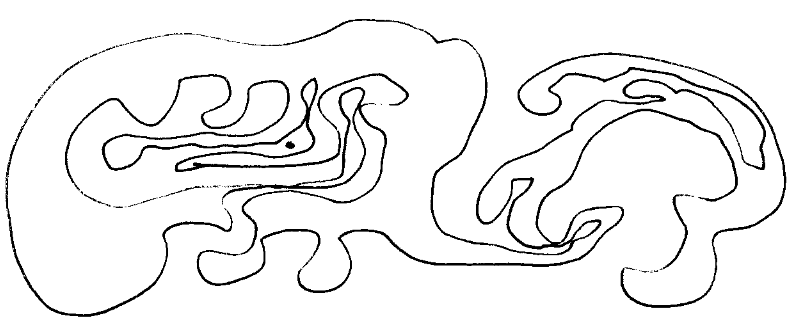
$$W_2(f, z) = W_2(f_1, z) + \ldots + W_2(f_n, z) \mod 2.$$

Since $F : B_i \to U$ is a diffeomorphism, by taking the ball small enough in D, we can ensure $W_2(f_i, z) = 1$ for all i, proving the Thm.

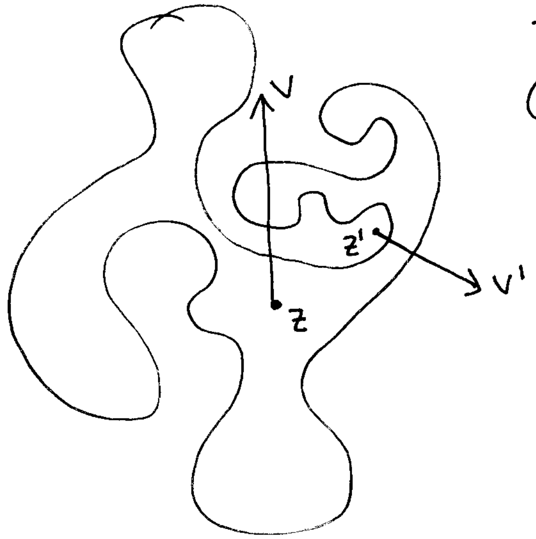
We can use this to prove:

Jordan-Brouwer Separation Theorem.

Given a compact, connected $n-1$ manifold $X \subset \mathbb{R}^n$, $\mathbb{R}^n - X$ ~~a~~ consists of two connected open sets, ~~the one~~ the "inside" I, whose closure is a compact n manifold with boundary X and the "outside" O.



The proof is a long and glorious homework assignment, so I'll give only the basic idea:



I = all z with $\omega_2(X, z) \neq 0$

O = all z' with $\omega_2(X, z') = 0$

Next time we'll prove

Borsuk-Ulam Theorem. Let $f: S^k \to \mathbb{R}^{k+1}$ be a map that avoids $0 \in \mathbb{R}^{k+1}$. Suppose f is odd

$$f(-x) = -f(x).$$

Then $\omega_2(f, 0) = 1$.