# Transversality and Orientation (III).

Given oriented vector spaces $V$ and $W$, we recall that the product orientation on $V \times W$ let

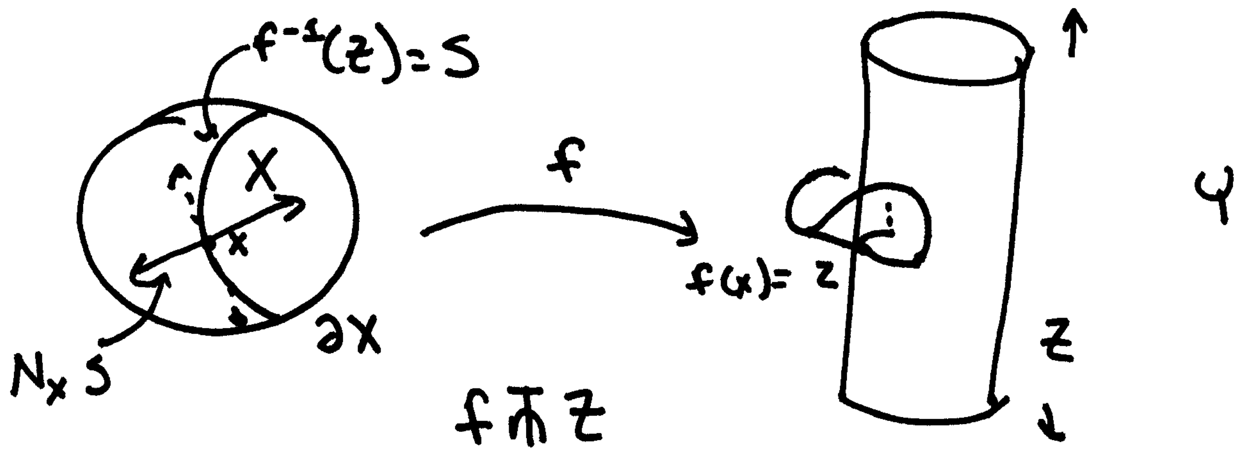
$$\mathrm{sign}(\alpha, \beta) = \mathrm{sign}(\alpha) \cdot \mathrm{sign}(\beta).$$

when $\alpha = (v_1, \ldots, v_n)$, $\beta = (w_1, \ldots, w_m)$ were bases for $V$ and $W$.

Observation. An orientation on any two of $V, W, V \times W$ determines an orientation on the third.

Proof. To determine the sign of $\beta$ (a basis for $W$) we choose any positive basis $\alpha$ for $V$ and compute the sign of $(\alpha, \beta)$ as a basis for $V \times W$.

## Orienting the preimage of a submanifold




$f \pitchfork Z$

$\partial f \pitchfork Z$

Observe that

$$T_x S = \text{preimage of } T_z Z \text{ under } df$$

let

$$N_x(S;X) = \text{orthogonal complement of } T_x S.$$

Now by construction

$$T_x S + N_x(S;X) = T_x X$$

so if we can orient $N_x(S;X)$, we can orient $T_x S$. (This is our goal.).

Now by transversality,

$$df_x\, T_x X + T_z Z = T_z Y.$$

~~but~~ ~~$df_x(T_x Z)$~~

recalling that $T_x X = T_x S + N_x(S;X)$, we see

$$df_x\, T_x X = df_x(T_x S) + df_x(N_x(S;X))$$

But $T_x S$ was the preimage of $T_z Z$, so

$$df\, T_x X + T_z Z = df\, N_x(S;X) + T_z Z = T_z Y.$$

Now $Z$ and $Y$ are oriented, so this determines an orientation on

$$df\, N_x(S;X).$$

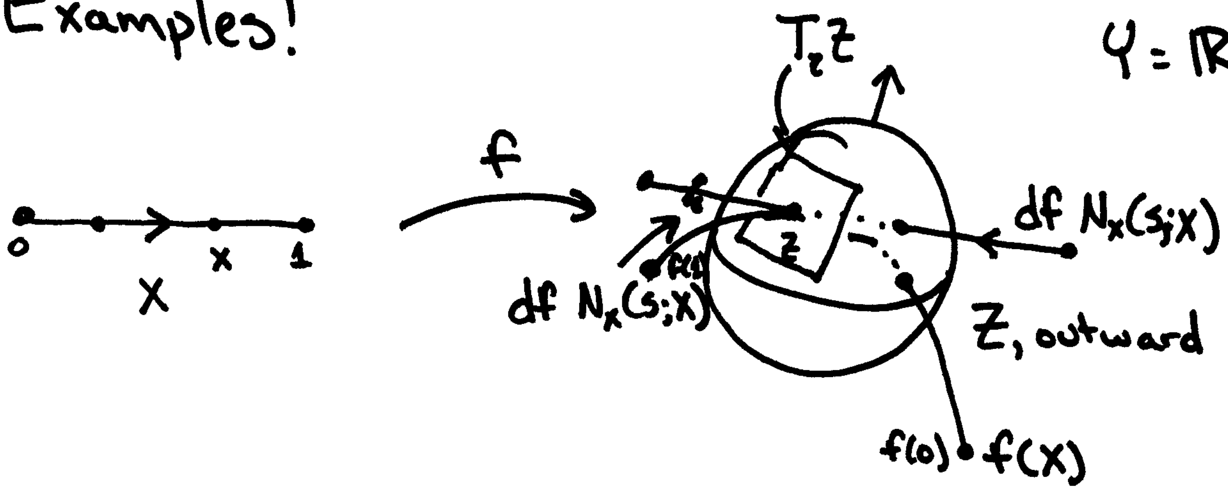
Yet $\ker df$ must have been ~~(except for 0)~~ contained in $T_x S$, so

$$df : N_x(S;X) \to df\, N_x(S;X)$$

is an isomorphism. Thus the orientation on $df\, N_x(S;X)$ induces an orientation on $N_x(S;X)$, and hence an orientation on $T_x S$.

# Examples!

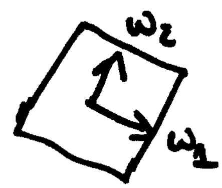
$Y = \mathbb{R}^3$, std.




$$T_x(X) = \underbrace{N_x(S;X)}_{1\text{ dim.}} + \underbrace{T_x S}_{0 \text{ dimensional}}$$

$$df\, N_x(S;X) + T_z Z = T_z Y$$

To compute the orientation on $df\, N_x(S;X)$, take a basis ~~elem~~, $\xleftarrow{v_1}$ and extend it by a positive basis for $T_z Z$




and ask whether

$\{v_1, \omega_1, \omega_2\}$ is a <u>positive</u> basis for $\mathbb{R}^3$, std.

Fact. To check std orientation of $\{u, v, w\}$ in $\mathbb{R}^3$ compute $(\vec{u} \times \vec{v}) \cdot \vec{\omega}$ and take sign.

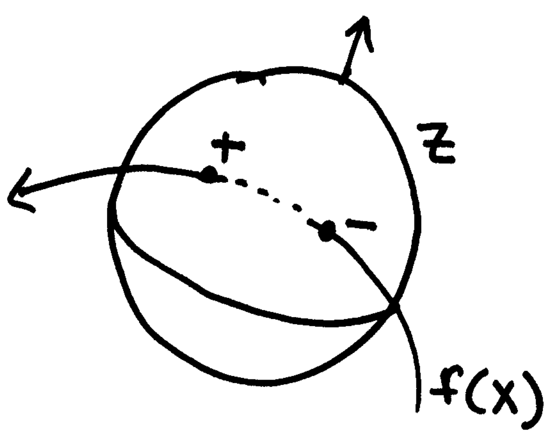
computing the sign, we see that it is +, so

$\nwarrow$ is a positive basis for $df\, N_x(S;X)$

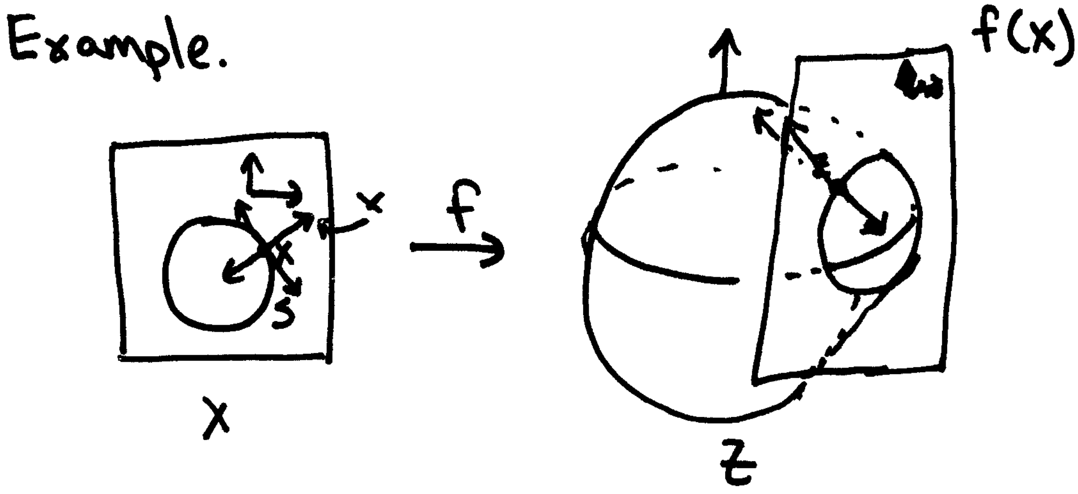
now we observe that

$$(df)^{-1}(\leftarrow) = \longrightarrow \text{ in } N_x(S;X)$$

which agrees with positive basis for $T_x X$, so $\text{sign}(\{x\}) = +$, and the orientation is positive (here) and negative (it turns out) at the other intersection.



Example.




$Y = \mathbb{R}^3$, std.

Compute preimage orientation of $S$ in $X$.

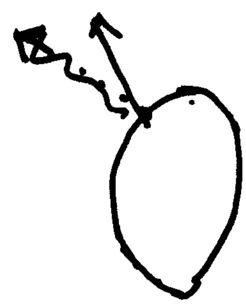
$$N_x(S;X) + T_x S = T_x X$$

We see that

$$\nearrow, \nwarrow \quad \text{is positive in } T_x X.$$

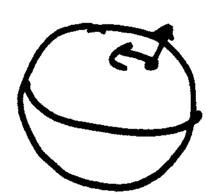
But what is the sign of $\nearrow$ in $N_x(S;X)$?

Well, $\nearrow$ maps to

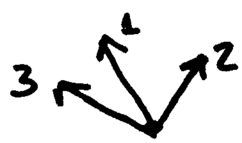


in $\mathbb{R}^3$,

and combined with a positive basis



for $T_z Z$, we get




which is + in $\mathbb{R}^3$.

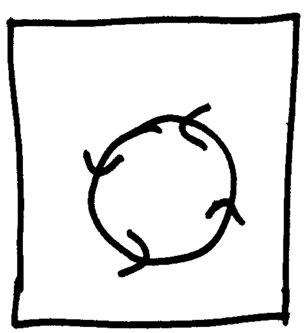
So that means that

$$\nwarrow \text{ is } + \text{ in } df\, N_x(S, X)$$

But then

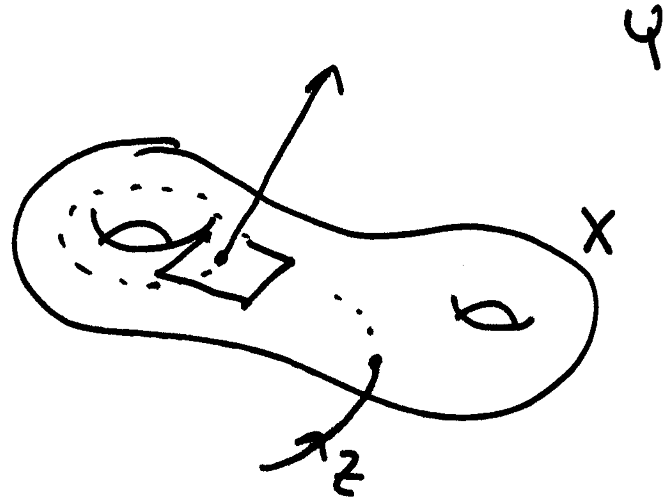
$$\nearrow = (df)^{-1}(\nwarrow)$$

is <u>positive</u> in $N_x(S, X)$, so $\nwarrow$ is positive in $T_x S$ and we get



as the preimage orientation.

Example.




If $X \pitchfork Z$ and they have complementary dimension in $Y$, then at $x \in X \cap Z$,

$$T_x X \times T_x Z = T_x Y$$

We claim $x$ is positively oriented in $X$ $\Leftrightarrow$ the orientation of $T_x Y$ is the product orientation.

(Homework.)