Probability.

<u>Definition</u>. If the value of a variable X is determined by chance, we call it a <u>random variable</u> or <u>r.v.</u> The <u>sample space</u> Ω is the set of possible ~~outcomes~~ values for X.

If the sample space is finite (or countable), the r.v. is <u>discrete</u>.

The elements of Ω are called <u>outcomes</u>. A subset $E \subset \Omega$ is called an <u>event</u>.

Example. A six-sided die is rolled. X is the number on the top.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$E = \{2, 4, 6\}$ is the <u>event</u> "the number rolled is even".

<u>Definition</u>. If X is an r.v. with sample space Ω, a <u>probability distribution function</u> (or <u>pdf</u>) for X is a function $m: \Omega \to \mathbb{R}$ so that

1) $m(\omega) \geq 0$ for each $\omega \in \Omega$
2) $\sum_{\omega \in \Omega} m(\omega) = 1$

<u>Definition</u>. For any event $E \subset \Omega$ we define the <u>probability</u> of E to be

$$P(E) = \sum_{\omega \in E} m(\omega).$$

<u>Example</u>. Let X be the sequence of heads and tails recorded in two (fair) coin flips.

$$\Omega = \{TT, TH, HT, HH\}$$

$m: \Omega \to \mathbb{R}$ is given by $m(\omega) = 1/4$.

E = "at least one head appears"

$= \{TH, HT, HH\}$

$P(E) = m(TH) + m(HT) + m(HH)$

$= 1/4 + 1/4 + 1/4 = 3/4$

We can describe events using set theory.

$E \vee F = E$ or $F = E \cup F = \{\omega \mid \omega \in E$ <u>or</u> $\omega \in F\}$.

$E \wedge F = E$ and $F = E \cap F = \{\omega \mid \omega \in E$ <u>and</u> $\omega \in F\}$.

E and not $F = E - F = \{\omega \mid \omega \in E$ and $\omega \notin F\}$.

$\neg E =$ not $E = \tilde{E} = \{\omega \mid \omega \notin E\}$.

It's easy to derive some properties of probabilities with these definitions.

Theorem. If X is an r.v. with sample space Ω and p.d.f. m, we have

1) $P(E) \geq 0$ for each $E \subset \Omega$

2) $P(\Omega) = 1$

3) If $E \subset F \subset \Omega$, then $P(E) \leq P(F)$

4) If $E \cap F = \emptyset$ (E, F are <u>disjoint</u>) then $P(E \cup F) = P(E) + P(F)$.

5) $P(\neg E) = 1 - P(E)$

Proof. This is straightforward. $\square$

In fact,

Theorem. If $E_1, \ldots, E_n$ is a collection of pairwise disjoint subsets of Ω, then

$$P(E_1 \cup \ldots \cup E_n) = \sum_i P(E_i)$$

This has a surprisingly useful consequence.

<u>Theorem</u>. Let $E_1, \dots, E_n$ be pairwise disjoint events with $\Omega = E_1 \cup \dots \cup E_n$. Then for any event E,

$$P(E) = \sum_{i=1}^{n} P(E \cap E_i)$$

<u>Proof</u>. We are going to use the <u>distributive</u> <u>law</u> for set theory.

$$E = E \cap \Omega = E \cap (E_1 \cup \dots \cup E_n)$$

$$= (E \cap E_1) \cup \dots \cup (E \cap E_n)$$

Since the E_i are pairwise disjoint, $E_i \cap E_j = \emptyset$. So

commutative law

$$(E \cap E_i) \cap (E \cap E_j) = E \cap E \cap E_i \cap E_j$$

$$= E \cap \emptyset = \emptyset$$

and the $E \cap E_i$ are pairwise disjoint too. Now the result follows from previous theorem. $\square$

Corollary. For any events E, F

$$P(E) = P(E \cap F) + P(E \cap \neg F).$$

There's another nice way to generalize:

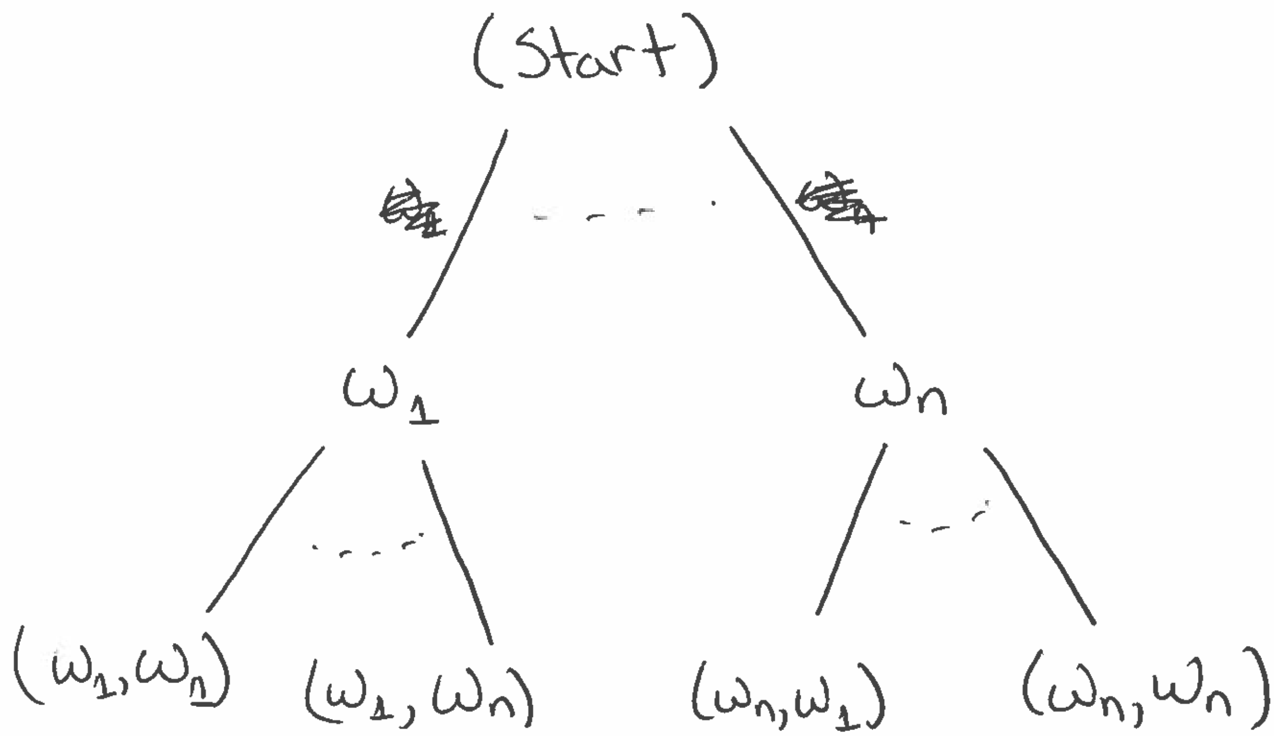
<u>Theorem</u>. For any events E, F, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Proof. Show each set of $m(\omega)$ is the same. $\square$

Tree diagrams.

When a sequence of values of an r.v. are taken, we can represent the combined experiment by a <u>tree</u> <u>diagram</u>



A path from the start (or root) of the tree to a leaf is an outcome of the combined experiment.

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Example. 3 coin flips.

E = at least one head

$\neg E$ = no heads

E = the sequence TT occurs.

Definition. The <u>uniform</u> <u>distribution</u> on Ω is the p.d.f. defined by

$$m(\omega) = \frac{1}{\#\Omega} = 1/\text{number of elements in } \Omega$$

for all ω.

Not every experiment has uniformly distributed outcomes! We have to <u>start</u> with a p.d.f.

Odds ratio.

If we are willing to bet that E occurs at "r to 1 odds" we mean that $P(E) = r\,P(\neg E)$.

Since $P(\neg E) = 1 - P(E)$, we can solve

$$P(E) = r\,(1 - P(E))$$

for $P(E)$ to get

$$P(E) = \frac{r}{1+r}$$

Example. Georgia has a 72% chance winning the SEC east....