Intersection # for submanifolds

We have defined

$I(f, Z)$ and proved that it is a homotopy invariant of f. But this is somewhat unsatisfying... what if Z were to change by homotopy?

We want to define

$$I(X, Z)$$

when X, Z are compact, boundaryless submanifolds of Y with complementary dimension (and everything is oriented).

Definition. $f \pitchfork g \iff df_x(T_xX) + dg_z(T_zZ) = T_yY$ when $f: X \to Y$, $g: Z \to Y$ take $x \mapsto y$, $z \mapsto y$.

Definition. The <u>local intersection #</u> of f, g at ~~at~~ (x, z) is $+1$ if the orientations on

$$df_x(T_xX) \quad \text{and} \quad dg_z(T_zZ)$$

add up (in that order) to the orientation on T_yY, and -1 otherwise.

We then have

Definition. $I(f,g) = \sum_{(x,z)\to y}$ local intersection # at (x,z).

We need to show that $I(f,g)$ is finite.

Idea: Consider

$$f \times g : X \times Z \to Y \times Y.$$

when $f(x) = g(z)$, then $(f\times g)(x,z)$ intersects the diagonal Δ of $Y \times Y$. So we want to show that

$$I(f\times g, \Delta) = I(f,g).$$

This is almost true, but a sign gets in the way.

Lemma. Let U, W be subspaces of V.

Then

$$U \oplus W = V \iff U \times W \oplus \Delta = V \times V$$

If U, W are oriented, and V has the sum ~~product~~ orientation, let Δ have the orientation induced from V by the isomorphism $V \to \Delta$.

The product orientation on $V \times V$ agrees with the sum orientation from $U \times W \oplus \Delta$ $\iff$ W is even dimensional.

In our case, this implies

Proposition. $f \pitchfork g \iff f \times g \pitchfork \Delta$ and

$$I(f, g) = (-1)^{\dim Z} I(f \times g, \Delta).$$

While $I(f,g)$ is only defined if $f \pitchfork g$, $I(f\times g, \Delta)$ is always defined, so we can use this to extend the def. of $I(f,g)$.

Consequences.

If $f_0 \simeq f_1$, $g_0 \simeq g_1$, then $I(f_0,g_0) = I(f_1,g_1)$.

If Z is a submanifold of Y and $i: Z \to Y$ the inclusion, then $I(f,i) = I(f,Z)$, for <u>any</u> map f.

$\deg(f)$ is well-defined.