Vector Fields on Manifolds

We have written vector fields on S^2, T^2 and so forth as if we were entirelly certain what they meant.

Here is a formal definition:

Definition. A <u>smooth vector field</u> is a smooth map $v: X \to TX$ so that the composition

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[dr, dashed] & TX \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

is the identity. Such a map is called a <u>section</u> of the tangent bundle.

In the case $X = \mathbb{R}^n$, a vector field is any smooth map $\vec{v}: \mathbb{R}^n \to \mathbb{R}^n$.

We define the index of an isolated zero $\vec{v}$ by taking a small ball S_ϵ centered at the zero

$\uparrow$

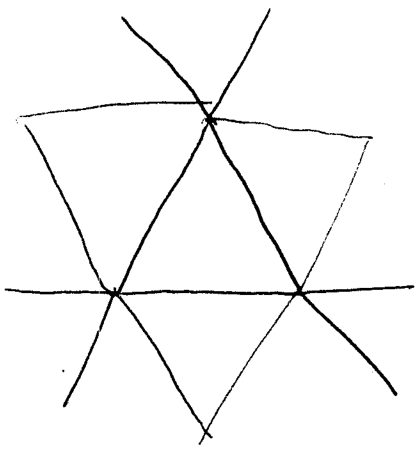
discussion of
Hopf index, Poincaré Hopf
theorem

$\downarrow$

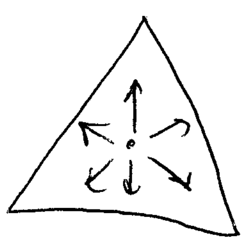
We are now ready to complete the circle of ideas by proving

$$\chi(S) = V - E + F$$

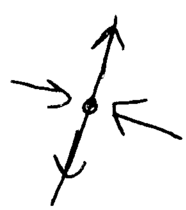
Suppose we have a 2-dimensional surface triangulated into vertices, edges and faces.



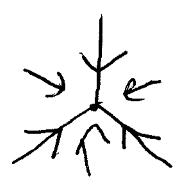
We can construct a special vector field on each face, edge, and vertex by pushing out from the center of each.



F = sum of #faces <u>sources</u>



− E = sum of #edges <u>saddles</u>



V = sum of # verts <u>sinks</u>

This proves

$$\chi(S) = V - E + F,$$

as previously claimed.