# Differentiation of forms

We have seen how to integrate forms. Further, we have seen that if $f$ is a diffeomorphism integration and pullback transform naturally into one another:

$$\int_X f^*\omega = \int_Y \omega, \quad \text{when } f: X \to Y.$$

We will now define a natural version of differentiation for forms on $\mathbb{R}^k$:

Recall: If $f$ is a 0-form, $df = \sum \frac{\partial f}{\partial x_j} dx_j$ is a 1-form.

Definition. If $\omega = \sum a_I \, dx_I$, then

$$d\omega = \sum da_I \wedge dx_I$$

We note that

Theorem.

(1). If $\omega$ is a p-form, $d\omega$ is a p+1 form.

(2). $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

(3). $d(\omega \wedge \Theta) = (d\omega) \wedge \Theta + (-1)^p \omega \wedge d\Theta$
if $\omega$ is a p-form

(4). $d(d\omega) = 0$.

(5). $d$ is the only operator so that $df = \langle -, \nabla f \rangle$ for functions with these properties.

Proof. (1), (2) are obvious.

(3). Well, if $\omega = \sum a_I \, dx_I$, $\Theta = \sum b_J \, dx_J$
then $\omega \wedge \Theta = \sum_{I,J} a_I b_J \, dx_I \wedge dx_J$, so

$$d(\omega \wedge \Theta) = \sum_{I,J} \cancel{a} \, b_J \, da_I \wedge dx_I \wedge dx_J$$

$$+ a_I \, db_J \wedge dx_I \wedge dx_J$$

$$= \sum (da_I \wedge dx_I) \wedge b_J \, dx_J + (-1)^p a_I \, dx_I \wedge db_J \wedge dx_J$$

$$= d\omega \wedge \Theta + (-1)^p \omega \wedge d\Theta.$$

To prove (4), we observe

$$d\omega = \sum da_I \wedge dx_I$$

$$= \sum_I \left( \sum_i \frac{\partial a_I}{\partial x_i}\, dx_i \right) \wedge dx_I$$

So

$$d(d\omega) = \sum_{I,i} d\left( \frac{\partial a_I}{\partial x_i} \right) \wedge dx_i \wedge dx_I$$

$$= \sum_{I,i,j} \frac{\partial^2 a_I}{\partial x_j \partial x_i}\, dx_j \wedge dx_i \wedge dx_I$$

But for each pair $(i,j)$ the corresponding pair $(j,i)$ also appears in the sum. Since $\frac{\partial^2 a_I}{\partial x_j \partial x_i} = \frac{\partial^2 a_I}{\partial x_i \partial x_j}$ we cancel these two by two.

(5). Suppose we had another operator $D$ with (1)-(4) true and $Df = df$ for functions. Observe

$$D(dx_I) = D(dx_{I_1} \wedge \dots \wedge dx_{I_p})$$
$$= \sum \pm dx_{I_1} \wedge \dots \wedge Ddx_k \wedge \dots \wedge dx_{I_p}$$

But $dx_k = Dx_k$, so $D\,dx_k = DDx_k = 0$, so

~~Now take~~

$$D(dx_I) = 0.$$

We now let $\omega$ be any p-form.

$$D\omega = \sum_I \left[ D(a_I) \wedge dx_I + a_I \underbrace{\cancel{D(dx_I)}}_{0} \right]$$
$$= \sum_I D(a_I) \wedge dx_I = \sum da_I \wedge dx_I = d\omega.$$

Corollary. Suppose $g: V \to U$ is a diffeomorphism, and $V \subset \mathbb{R}^K$, $U \subset \mathbb{R}^K$. Then for any $\omega$ on $U$,

$$d(g^*\omega) = g^*(d\omega).$$

Proof. It is easy to see

$$D = (g^{-1})^* \circ d \circ g^*$$

obeys the properties above (2)-(4). We already know for functions that

$$d(g^* f) = g^*(df)$$

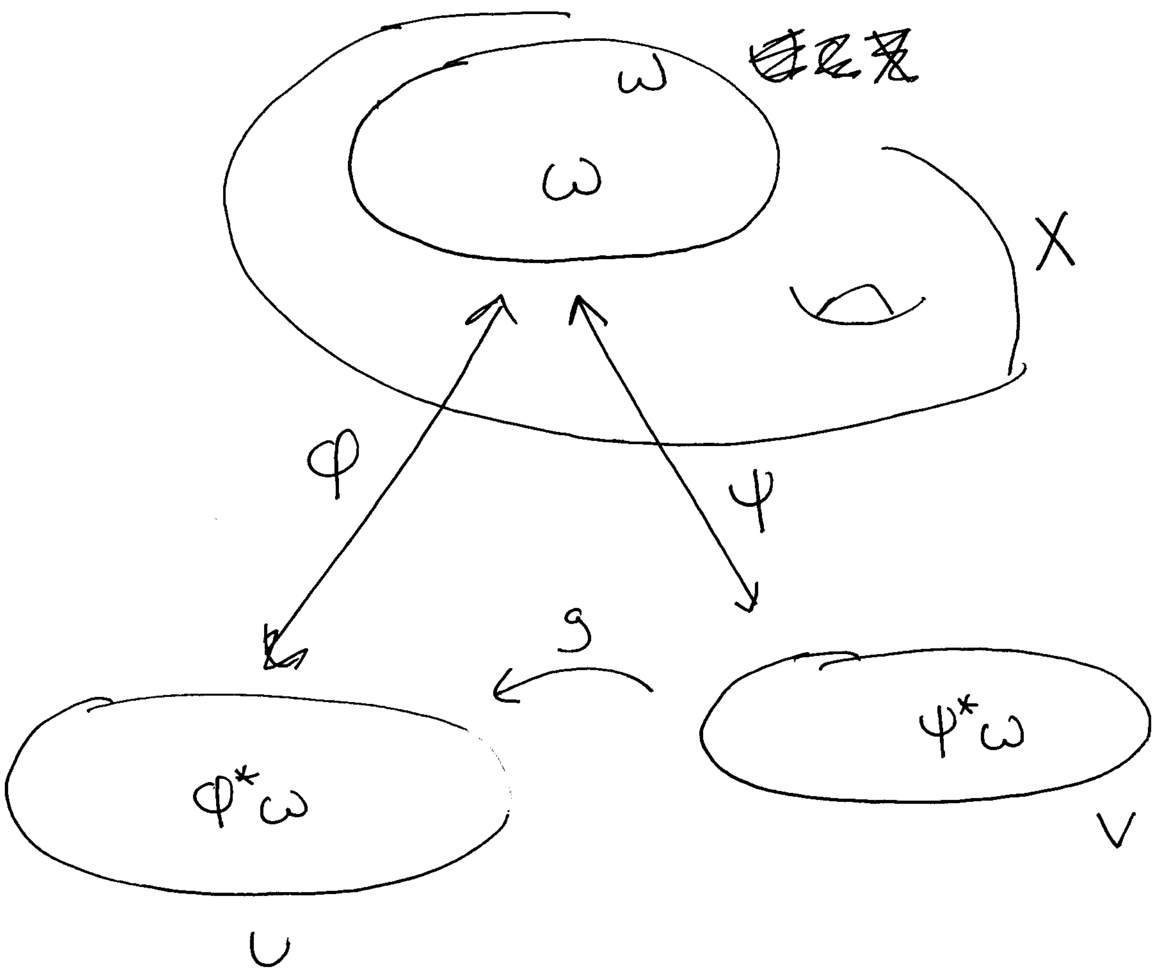
so $D = d$, or

$$\cancel{d}\,(g^{-1})^* \circ d \circ g^* = d$$

or

$$d \circ g^* = g^* \circ d.$$

We will use this transformation property to define $d$ for forms on manifolds.




For ~~ex~~ a form ~~on~~ $\omega$ on $W$, we let

$$d\omega = (\varphi^{-1})^* d(\varphi^*\omega)$$

Now if $g = \varphi^{-1} \circ \psi$, we already know

$$g^*(d(\varphi^*\omega)) = d(g^*(\varphi^*\omega)) = d(\psi^*\omega)$$

so

$$(\psi^{-1})^* d(\psi^* \omega) = (\psi^{-1})^* g^* d(\varphi^* \omega)$$
$$= (g \circ \psi^{-1})^* d(\varphi^* \omega)$$
$$= (\varphi^{-1})^* d(\varphi^* \omega).$$

so (as expected) this doesn't depend on the choice of coordinates.

All of the previous properties hold for $d$ on manifolds. So we can show

Theorem. Let $g: Y \to X$ be any smooth map of manifolds (which may have boundary). Then for any $\omega$ on $X$,

$$d(g^* \omega) = g^*(d\omega).$$

Observation: We know this is true for any 0-form $f$. Further, for any $df$,

$$d(g^* df) = d(d\, g^* f) = 0$$

and

$$g^*(d(df)) = g^*(0) = 0.$$

Now suppose the theorem holds for $\omega$ and $\theta$. It is easy to compute

$$\begin{aligned}
d(g^*(\omega \wedge \theta)) &= d(g^*\omega \wedge g^*\theta) \\
&= d(g^*\omega) \wedge g^*\theta + (-1)^p g^*\omega \wedge d g^*\theta \\
&= g^*(d\omega) \wedge g^*\theta + (-1)^p g^*\omega \wedge g^* d\theta \\
&= g^*(d\omega \wedge \theta + (-1)^p \omega \wedge d\theta) \\
&= g^*(d(\omega \wedge \theta)).
\end{aligned}$$

so it holds for $\omega \wedge \theta$ as well.

However, <u>every</u> $\omega = \sum a_I dx_I$ so since the theorem holds for the $a_I$ and ~~$dx_I$~~, $dx_i$, we're done.

Examples. In $\mathbb{R}^3$,

f is a function

0

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

$\boxed{df \text{ is grad } f.}$

1. $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ is a vector field. Then

$$d\omega = df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3$$

$$= \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3$$

$$+ \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_3 \wedge dx_1$$

$$+ \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

$\boxed{d\omega \text{ is curl } \omega}$

$$\omega = f_1 \, dx_2 \wedge dx_3 + f_2 \, dx_3 \wedge dx_1 + f_3 \, dx_1 \wedge dx_3$$

(2)
$$d\omega = \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3.$$

$$= \left( \operatorname{div} (f_1, f_2, f_3) \right) dx_1 \wedge \dots \wedge dx_3.$$

$$\boxed{d\omega \text{ is } \operatorname{div} \omega.}$$

This means (satisfying, isn't it?) that

grad, curl, and div
are all really d.

And explains the facts

curl (grad f) = 0

div (curl f) = 0

in a natural way.