Math 4250/6250

Key ideas of course:

Math started with geometry.
Shapes are all around you—but they aren’t accidental. Why?
Linear algebra, differential equations and calculos are surprisingly powerful tools for understanding shape.
Definition. The **dot product** of vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is given by
\[
\langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i
\]

Definition. The **length** (or **norm**) of a vector $\vec{v} \in \mathbb{R}^n$ is given by
\[
\| \vec{v} \| = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{\langle \vec{v}, \vec{v} \rangle}
\]

**Theorem (Cauchy-Schwartz)** For any vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, $|\langle \vec{v}, \vec{w} \rangle| \leq \| \vec{v} \| \| \vec{w} \|$ with equality $\iff$ one vector is a scalar multiple of the other.

Definition. The **angle** between $\vec{v}, \vec{w} \in \mathbb{R}^n$ is defined by $\cos \Theta = \frac{\langle \vec{v}, \vec{w} \rangle}{\| \vec{v} \| \| \vec{w} \|}$, $\Theta \in [0, \pi]$. 
Proof (of C.5). If \( \hat{w} = \hat{0} \), we're done. If \( \hat{w} \neq \hat{0} \), consider
\[
g(t) = \| \hat{x} + t \hat{y} \|^2
= \langle \hat{x} + t \hat{y}, \hat{x} + t \hat{y} \rangle
= \langle \hat{x}, \hat{x} \rangle + 2t \langle \hat{x}, \hat{y} \rangle + t^2 \langle \hat{y}, \hat{y} \rangle
\]
We can find the minimum of this quadratic function of \( t \) by differentiating
\[
g'(t) = 2\langle \hat{x}, \hat{y} \rangle + 2t \langle \hat{y}, \hat{y} \rangle
\]
and solving \( g'(t) = 0 \) to get
\[
t_0 = -\frac{\langle \hat{x}, \hat{y} \rangle}{\langle \hat{y}, \hat{y} \rangle}
\]
Now \( g(t) \geq 0 \) for all \( t \), so we
Know that
\[
g(t_0) = \langle \dot{x}, \dot{x} \rangle - 2\frac{\langle \dot{x}, \dot{y} \rangle^2}{\langle \dot{y}, \dot{y} \rangle} + \frac{\langle \dot{y}, \dot{y} \rangle^2}{\langle \dot{y}, \dot{y} \rangle} \]

\[
= \langle \dot{x}, \dot{x} \rangle - \frac{\langle \dot{x}, \dot{y} \rangle^2}{\langle \dot{y}, \dot{y} \rangle} \geq 0
\]

Rearranging, we get
\[
\langle \dot{x}, \dot{x} \rangle \langle \dot{y}, \dot{y} \rangle \geq \langle \dot{x}, \dot{y} \rangle^2
\]
or (taking square roots)
\[
\|\dot{x}\| \|\dot{y}\| \geq |\langle \dot{x}, \dot{y} \rangle|
\]
We have equality \(\iff g(t_0) = 0\), in which case \(\dot{x} = -t_0 \dot{y}\) and \(\dot{x}\) is a scalar multiple of \(\dot{y}\). \(\Box\)
We are going to use this to prove Proposition. (The triangle inequality)
For any vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,
$$
\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.
$$

with equality $\iff$ $\mathbf{v}$ is a positive scalar multiple of $\mathbf{w}$.

Proof. We compute

$$
\|\mathbf{v} + \mathbf{w}\|^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle
= \langle \mathbf{v}, \mathbf{v}\rangle + 2\langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle
\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2
= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2.
$$

Cauchy-Schwarz! \hfill \Box

Now we switch from “geometry” to “differential geometry” by proving an integrated version of this inequality.
Theorem. If $\dot{\mathbf{x}}(t) : [a, b] \to \mathbb{R}^n$ is a continuous vector-valued function, then

$$\| \int_a^b \dot{\mathbf{x}}(t) \, dt \| \leq \int_a^b \| \dot{\mathbf{x}}(t) \| \, dt.$$  
with equality $\iff \frac{\dot{\mathbf{x}}(t)}{\| \dot{\mathbf{x}}(t) \|}$ is constant.

Proof. Let $\mathbf{v} \in \mathbb{R}^n$ be any vector with $\| \mathbf{v} \| = 1$. Then for each $t \in [a, b]$,

$$\| \dot{\mathbf{x}}(t) \| = \| \dot{\mathbf{x}}(t) \| \| \mathbf{v} \|$$

Cauchy-Schwarz $\implies \quad \langle \dot{\mathbf{x}}(t), \mathbf{v} \rangle$

$$\int_a^b \| \dot{\mathbf{x}}(t) \| \, dt \geq \int_a^b \langle \dot{\mathbf{x}}(t), \mathbf{v} \rangle \, dt$$

$$= \int_a^b \sum_{i=1}^n \alpha_i(t) v_i \, dt$$

ith component of $\dot{\mathbf{x}}(t)$
\[ n \leq \sum_{i=1}^{n} v_i \int_{a}^{b} \alpha_i(t) \, dt \]

\[ = \langle \vec{v}, \int_{a}^{b} \vec{\alpha}(t) \, dt \rangle \]

Now this is true for all \( \vec{v} \), so wlog we may assume that \( \vec{v} \) is a scalar multiple of \( \int_{a}^{b} \vec{\alpha}(t) \, dt \).

By Cauchy-Schwarz, in that case,

\[ = \| \vec{v} \| \| \int_{a}^{b} \vec{\alpha}(t) \, dt \| \]

\[ = \| \int_{a}^{b} \vec{\alpha}(t) \, dt \| . \]

Equality holds if \( \vec{\alpha}(t) \) is a positive scalar multiple of \( \vec{v} \) for each \( t \), or \( \frac{\vec{\alpha}(t)}{\| \vec{\alpha}(t) \|} = \vec{v} \) for all \( t \). \( \square \)
We start by studying curves.

Definition. A function \( \vec{x}: \mathbb{R} \rightarrow \mathbb{R}^n \) is called a parametrized curve. We write \( \vec{x}(t) = (x_1(t), \ldots, x_n(t)) \).

Recall that the derivative of \( \vec{x} \),

\[
\vec{x}'(t) = (x_1'(t), \ldots, x_n'(t))
\]
is also a vector valued function.

Definition. The length \( \| \vec{x}'(t) \| \) is called the speed of \( \vec{x}(t) \). The vector \( \vec{x}'(t) \) is called the velocity vector of \( \vec{x}(t) \).

Definition. If \( \vec{x}: [a, b] \rightarrow \mathbb{R}^n \), the length of the curve is

\[
\int_a^b \| \vec{x}'(t) \| \, dt
\]
We are now ready to prove our first main theorem!

Theorem. If $\vec{v}, \vec{w} \in \mathbb{R}^n$ are any two points, a shortest differentiable curve $\vec{\alpha} : [0,1] \to \mathbb{R}^n$ with $\vec{\alpha}(0) = \vec{v}$ and $\vec{\alpha}(1) = \vec{w}$ is the straight line

$$\vec{\alpha}(t) = \vec{v} + t (\vec{w} - \vec{v})$$

which has length $\|\vec{v} - \vec{w}\|$. 
Proof. If \( \dot{\mathbf{a}} : [0,1] \to \mathbb{R}^n \) is diff.,
\[
\mathbf{a}(1) - \mathbf{a}(0) = \int_0^1 \dot{\mathbf{a}}'(t) \, dt
\]
and so
\[
\| \mathbf{a}(1) - \mathbf{a}(0) \| = \| \int_0^1 \dot{\mathbf{a}}'(t) \, dt \|
\]
\[
= \int_0^1 \| \dot{\mathbf{a}}'(t) \| \, dt = \text{length of } \dot{\mathbf{a}}.
\]
Thus the length of \( \dot{\mathbf{a}} \) is at least
\[
\| \mathbf{a}(1) - \mathbf{a}(0) \| = \| \mathbf{\hat{w}} - \mathbf{\hat{v}} \|. \text{ If}
\]
\[
\mathbf{a}(t) = \mathbf{\hat{v}} + t(\mathbf{\hat{w}} - \mathbf{\hat{v}})
\]
then \( \dot{\mathbf{a}}'(t) = \mathbf{\hat{w}} - \mathbf{\hat{v}} \), and
\[
\| \dot{\mathbf{a}}'(t) \| = \| \mathbf{\hat{w}} - \mathbf{\hat{v}} \|, \text{ so the length is}
\]
exactly
\[
\int_0^1 \| \mathbf{\hat{w}} - \mathbf{\hat{v}} \| \, dt = \| \mathbf{\hat{w}} - \mathbf{\hat{v}} \|. \quad \square
\]
Notice that we haven’t proved that the line is the unique length minimizing curve. (That will take more theory!)

Now suppose we have two curves.

Proposition. If \( \dot{\alpha}, \dot{\beta} : \mathbb{R} \to \mathbb{R}^n \) are differentiable functions, then \( \langle \dot{\alpha}(t), \dot{\beta}(t) \rangle \) is differentiable and

\[
\frac{d}{dt} \langle \dot{\alpha}(t), \dot{\beta}(t) \rangle = \langle \dot{\alpha}'(t), \dot{\beta}(t) \rangle + \langle \dot{\alpha}(t), \dot{\beta}'(t) \rangle.
\]

Proof. (homework)
Proposition. If $\mathbf{\alpha}(t)$ is a vector valued function so that $\|\mathbf{\alpha}(t)\| = 1$, then 
\[ \langle \mathbf{\dot{\alpha}}(t), \mathbf{\alpha}(t) \rangle = 0. \]

Proof. If $\|\mathbf{\dot{\alpha}}(t)\| = 1$, then $\|\mathbf{\dot{\alpha}}(t)\|^2 = 1$, or $\langle \mathbf{\dot{\alpha}}(t), \mathbf{\dot{\alpha}}(t) \rangle = 1$. Differentiating both sides,
\[ 0 = \frac{d}{dt} \langle \mathbf{\dot{\alpha}}(t), \mathbf{\dot{\alpha}}(t) \rangle = \langle \mathbf{\ddot{\alpha}}(t), \mathbf{\dot{\alpha}}(t) \rangle + \langle \mathbf{\dot{\alpha}}(t), \mathbf{\ddot{\alpha}}(t) \rangle 
= 2\langle \mathbf{\dot{\alpha}}(t), \mathbf{\dot{\alpha}}(t) \rangle. \]

We now recall (for vectors in $\mathbb{R}^3$)

Definition. If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, the cross product
\[ \mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1). \]
Properties.

1) The cross product is bilinear.
2) \( \mathbf{v} \times \mathbf{w} \) is orthogonal to \( \mathbf{v} \) and \( \mathbf{w} \).
3) \( \| \mathbf{v} \times \mathbf{w} \| = \| \mathbf{v} \| \| \mathbf{w} \| \sin \theta \),
   where \( \theta \) is the angle between \( \mathbf{v} \) and \( \mathbf{w} \).
4) \( \mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \).

We will often use

Definition. If \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \), the triple product
is \( \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle \).

Properties.

1) The triple product is trilinear.
2) \( \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = \text{det} \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \)
3) \( \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = -\langle \mathbf{v}, \mathbf{u} \times \mathbf{w} \rangle = -\langle \mathbf{w}, \mathbf{u} \times \mathbf{v} \rangle \\
   = -\langle \mathbf{w}, \mathbf{v} \times \mathbf{u} \rangle \)
Proposition. If $\mathbf{\dot{a}}(t)$, $\mathbf{\dot{b}}(t)$ are vector valued functions in $\mathbb{R}^3$,

$$\frac{d}{dt} \mathbf{\dot{a}}(t) \times \mathbf{\dot{b}}(t) = \mathbf{\dot{a}}'(t) \times \mathbf{\dot{b}}(t) + \mathbf{\dot{a}}(t) \times \mathbf{\dot{b}}'(t).$$

Proof. (Homework)  

note the order

This gives us our first set of tools to think about curve geometry. Next time, we'll practice constructing some curves to play with.