Math 4250/6250

The idea of this course is to

Combine

multivariable calculus + linear algebra $\longrightarrow$ insight about shapes

The approach is basically a sampler of topics which illustrate different techniques and ideas.

They are connected by theme, but basically independent, so you'll have several chances to get back on board if you lose your way.

We start by studying curves.

Definition. A function $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ is called a parametrized curve. We write $\vec{\alpha}(t) = (\alpha_1(t), \dots, \alpha_n(t))$.

Recall that the derivative of $\vec{\alpha}$,

$$\vec{\alpha}'(t) = (\alpha_1'(t), \dots, \alpha_n'(t))$$

is also a vector valued function.

Definition. The length $\|\vec{\alpha}'(t)\|$ is called the speed of $\vec{\alpha}(t)$. The vector $\vec{\alpha}'(t)$ is called the velocity vector of $\vec{\alpha}(t)$.

As in single-variable calculus,

$$\int_a^b \vec{\alpha}'(t)\,dt = \underbrace{\vec{\alpha}(b) - \vec{\alpha}(a)}_{\text{the displacement vector}}$$

while

$$\int_a^b \|\vec{\alpha}'(t)\|\,dt = \text{distance traveled.}$$

Proposition. For any vector-valued function $\vec{\beta}(t)$, we have

$$\int_a^b \|\vec{\beta}(t)\|\,dt \geq \left\| \int_a^b \vec{\beta}(t)\,dt \right\|$$

If $\vec{\beta}(t) = \vec{\alpha}'(t)$, this is immediately believable: If you drive at 30 mph for one hour, you cover 30 miles

$$\left(\int_a^{\cancel{b}\,1} \|\vec{\alpha}'(t)\|\,dt = \int_0^1 30\,dt = 30 \right)$$

and the distance from your starting position to your ending position is at most 30 miles.

$$\left(\left\| \int_0^1 \alpha'(t)\,dt \right\| = \|\vec{\alpha}(1) - \vec{\alpha}(0)\| \leq 30. \right)$$

We're going to prove this very slowly and carefully, using it as an opportunity to remember some facts about vectors and linear algebra.

Definition. The dot product of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is given by

$$\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle = v_1 w_1 + \dots + v_n w_n.$$

We prefer the notation $\langle \vec{v}, \vec{w} \rangle$ to $\vec{v} \cdot \vec{w}$ for this class.

Recall that

1) $\langle \vec{v}, \vec{w} \rangle$ is bilinear.

$$\langle \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2, \mu_1 \vec{w}_1 + \mu_2 \vec{w}_2 \rangle$$

$$= \lambda_1 \mu_1 \langle \vec{v}_1, \vec{w}_1 \rangle + \lambda_1 \mu_2 \langle \vec{v}_1, \vec{w}_2 \rangle$$

$$\lambda_2 \mu_1 \langle \vec{v}_2, \vec{w}_1 \rangle + \lambda_2 \mu_2 \langle \vec{v}_2, \vec{w}_2 \rangle.$$

2) $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

3) If θ is the angle between $\vec{v}, \vec{w}$

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \, \|\vec{w}\| \cos\theta = \langle \vec{w}, \vec{v} \rangle$$

Proof. Suppose $\vec{v}$ is any vector in $\mathbb{R}^n$ with $\|\vec{v}\| = 1$. Then for each t,

$$\|\vec{\beta}(t)\| \geq \|\vec{\beta}(t)\| \, \|\vec{v}\| \cos\theta(t)$$

$$= \langle \vec{\beta}(t), \vec{v} \rangle$$

if $\theta(t)$ is the angle between $\vec{\beta}(t)$ and $\vec{v}$

so

$$\int_a^b \|\vec{\beta}(t)\|\,dt \geq \int_a^b \langle \vec{\beta}(t), \vec{v} \rangle\,dt.$$

Now $\vec{v} = (v_1, \dots, v_n)$ is constant, so

$$\int_a^b \langle \vec{\beta}(t), \vec{v} \rangle\,dt = \int_a^b v_1\beta_1(t) + \dots + v_n\beta_n(t)\,dt$$

$$= v_1\int_a^b \beta_1(t)\,dt + \dots + v_n\int_a^b \beta_n(t)\,dt$$

$$= \left\langle \vec{v}, \int_a^b \vec{\beta}(t)\,dt \right\rangle.$$

If φ is the angle between $\vec{v}$ and $\int_a^b \beta(t)\,dt$ we have

$$\left\langle \vec{v}, \int_a^b \vec{\beta}(t)\,dt \right\rangle = \underbrace{\|\vec{v}\|}_{1} \left\| \int_a^b \vec{\beta}(t)\,dt \right\| \cos\varphi$$

Choosing $\vec{v}$ so that $\varphi = 0$, we have

$$\int_a^b \|\vec{\beta}(t)\|\,dt \geq \left\| \int_a^b \vec{\beta}(t)\,dt \right\|. \qquad \square$$

We note that for vectors $\vec{\beta}(t) \in \mathbb{R}^1$,

$$\|\vec{\beta}(t)\| = \sqrt{\beta_1^2(t)} = |\beta_1(t)|$$

and we have proved

$$\int_a^b |\beta(t)|\,dt \geq \left| \int_a^b \beta(t)\,dt \right|.$$

Proposition. If $\vec{\alpha}(t)$ and $\vec{\beta}(t)$ are vector valued functions $\mathbb{R} \to \mathbb{R}^n$ then

$$\frac{d}{dt}\langle \vec{\alpha}(t), \vec{\beta}(t) \rangle = \langle \vec{\alpha}'(t), \beta(t) \rangle + \langle \vec{\alpha}(t), \vec{\beta}'(t) \rangle.$$

Proof. Homework.

We can now recall

Definition. $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal if the angle between them is $\pi/2$. or, Equivalently, if $\langle \vec{v}, \vec{w} \rangle = 0$.

Proposition. If $\vec{\alpha}(t)$ is a vector valued function so that $\|\vec{\alpha}(t)\| \equiv 1$, then $\langle \vec{\alpha}'(t), \vec{\alpha}(t) \rangle = 0$.

Proof. If $\|\vec{\alpha}(t)\| \equiv 1$, then $\|\vec{\alpha}(t)\|^2 \equiv 1$, or $\langle \vec{\alpha}(t), \vec{\alpha}(t) \rangle \equiv 1$. Differentiating both sides,

$$0 = \frac{d}{dt} \langle \vec{\alpha}(t), \vec{\alpha}(t) \rangle = \langle \vec{\alpha}'(t), \vec{\alpha}(t) \rangle + \langle \vec{\alpha}(t), \vec{\alpha}'(t) \rangle$$
$$= 2 \langle \vec{\alpha}'(t), \vec{\alpha}(t) \rangle . \qquad \boxtimes$$

We now recall (for vectors in $\mathbb{R}^3$)

Definition. If $\vec{v}, \vec{w} \in \mathbb{R}^3$, the <u>cross product</u>

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

Properties.

1) The cross product is bilinear.

2) $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$

3) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$,
where θ is the angle between $\vec{v}$ and $\vec{w}$.

4) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

We will often use

Definition. If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, the <u>triple product</u> is $\langle \vec{u}, \vec{v} \times \vec{w} \rangle$.

Properties.

1) The triple product is trilinear.

2) $\langle \vec{u}, \vec{v} \times \vec{w} \rangle = \det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$

3) $\langle \vec{u}, \vec{v} \times \vec{w} \rangle = -\langle \vec{v}, \vec{u} \times \vec{w} \rangle = -\langle \vec{u}, \vec{w} \times \vec{v} \rangle$
$= -\langle \vec{w}, \vec{v} \times \vec{u} \rangle$

Proposition. If $\vec{\alpha}(t)$, $\vec{\beta}(t)$ are vector valued functions in $\mathbb{R}^3$,

$$\frac{d}{dt}\vec{\alpha}(t)\times\vec{\beta}(t)=\vec{\alpha}'(t)\times\vec{\beta}(t)+\vec{\alpha}(t)\times\vec{\beta}'(t).$$

Proof. (Homework)