Advanced Error Analysis of the Trapezoid Rule.

We start with a cod observation from our proof of the error formula

$$\int_a^b f(x)\,dx - T(n) = -\frac{(b-a)^2}{12} f''(\xi) h^2$$

Recall that we really proved

$$\int_a^b f(x)\,dx - T(n) = -\frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i)$$

where $\xi_i \in [x_i, x_{i+1}]$. (Page 6 of "Numerical integration")

But

$$-\frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i) = -\frac{h^2}{12} \sum_{i=0}^{n-1} h f''(\xi_i)$$

So if this error is $E(n)$, we have the <u>asymptotic</u> error estimate

$$\lim_{n\to\infty} \frac{E(n)}{h^2} = -\frac{1}{12} \lim_{n\to\infty} \sum_{i=0}^{n-1} h f''(\xi_i)$$

$$= -\frac{1}{12} \int_a^b f''(x)\,dx = -\frac{1}{12}\, f'(b) - f'(a).$$

We now need a new trick. We have gotten a lot of mileage out of Taylor's theorem with the Lagrange error term:

$$f(x+h) = f(x) + f'(x)h + \dots + \frac{f^{(n)}(x)h^n}{n!} + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

and ξ is an unknown point in $(x, x+h)$. But we can also write

$$R_n(x) = \int_x^{x+h} \frac{f^{(n+1)}(t)}{n!}((x+h)-t)^n \, dt.$$

This is the integral form of the remainder term.

We can use this to get error formulae for the trapezoid rule w/o those (nasty) evaluations at unknown pts.

Here's how it goes: Take expansion of f at a, and get

$$f(x) = p_1(x) + R_1(x)$$

$$= f(a) + (x-a)f'(a) + \int_a^x (x-t) f''(t)\,dt.$$

Now if $E(n,f)$ is the error in the trapezoid rule for f with n points, we can show (Acton, p.252) that

$$E(1, f+g) = E(1,f) + E(1,g).$$

So

$$E(1,f) = E_1(1, p_1) + E_1(1, R_1)$$

But p_1 is a linear function, so

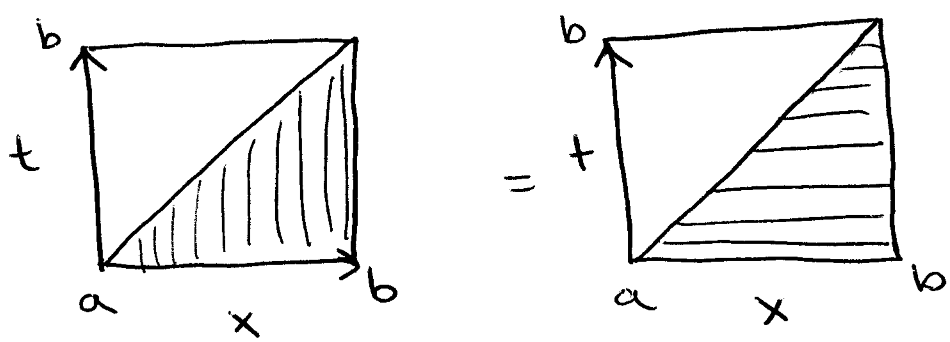
$$E(1,f) = E(1, R_1).$$

Now over $[a,b]$

$$E(1,R_1) = \int_a^b R_1(x)\,dx - \left(\frac{b-a}{2}\right)[R_2(a)-R_2(b)]$$

$$= \int_a^b \int_a^x (x-t) f''(t)\,dt\,dx - \left(\frac{b-a}{2}\right)\int_a^b (b-t) f''(t)\,dt.$$

If we consider the first integral, we are integrating over



So we can write

$$= \int_a^b \int_t^b (x-t) f''(t)\,dx\,dt - \left(\frac{b-a}{2}\right)\int_a^b (b-t) f''(t)\,dt$$

$$= \int_a^b f''(t)\left[\int_t^b (x-t)\,dx - \left(\frac{b-a}{2}\right)(b-t)\right]dt$$

Now

$$\int_t^b (x-t)\,dx = \left.\frac{(x-t)^2}{2}\right|_{x=t}^{b}$$

$$= \frac{(b-t)^2}{2}$$

So we have

$$= \int_a^b f''(t)\,(b-t)\left[\frac{(b-t)}{2} - \frac{(b-a)}{2}\right] dt$$

$$= \frac{1}{2}\int_a^b f''(t)\,(b-t)(a-t)\,dt$$

$$= \frac{1}{2}\int_a^b f''(t)\,(t-a)(t-b)\,dt.$$

If we define

$$K(t) = \frac{1}{2}(t-x_i)(t-x_{i+1}), \quad x_i \le t \le x_{i+1}$$

then it's not surprising that the general error term comes out to be

$$E(n,f) = \int_a^b K(t)\, f''(t)\, dt.$$

<u>Definition.</u> $K(t)$ is called the <u>Peano Kernel</u> for the trapezoid rule, and this is the Peano Kernel formulation of the error estimate for the trapezoid rule.

Since

$$\max_{t \in [a,b]} K(t) = \frac{h^2}{8},$$

we get

$$|E(n,f)| \leq \frac{h^2}{8} \int_a^b |f''(t)|\, dt.$$

sneaky use of Jensen's inequality

<Peano_Kernel.nb>

We now try out our error formulae in an example

<trapezoid_rule.nb>.

This demands explanation! We now develop a better error bound.

We need some background...

We now introduce the Bernoulli polynomials. These are defined to be

$$\frac{t(e^{xt}-1)}{e^t-1} = \sum_{j=1}^{\infty} B_j(x)\frac{t^j}{j!}$$

It's not hard to see

$$B_0 = 1 \qquad B_1 = x \qquad B_2 = x^2 - x$$

$$B_3 = x^3 - \frac{3x^2}{2} + \frac{x}{2} \qquad B_4 = x^2(1-x)^2$$

For $K \geq 1$, $B(0)=0$.

The Bernoulli numbers are given by

$$\frac{t}{e^t-1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}$$

<Bernoulli-polynomials.nb>

We are now going to derive a very clever integral estimate for the error in the trapezoid rule (this is from Atkinson, p. 285).

Theorem. If $m \geq 0$, $n \geq 1$, $h = \frac{b-a}{n}$, $x_j = a + jh$ & (for $j \in 0, \ldots, n$), and f is C^{2m+2} on $[a,b]$. Then the error in the n-interval trapezoid rule is given by

$$E_n(f) = -\sum_{i=1}^{m} \frac{B_{2i}}{(2i)!} h^{2i} \left[f^{(2i)}(b) - f^{(2i-1)}(a) \right]$$

$$+ \frac{h^{2m+2}}{(2m+2)!} \int_a^b \bar{B}_{2m+2}\left(\frac{x-a}{h} \right) f^{(2m+2)}(x)\, dx.$$

Here we go. We know

$$E(1,f) = \int_0^h f(x)\,dx - \frac{h}{2}[f(0)+f(h)]$$
$$= \frac{1}{2}\int_0^h f''(x)\,x(x-h)\,dx$$

(the Peano Kernel error term from last class)

We have also the asymptotic formula

$$E_1(1,f) \approx -\frac{h^2}{12}[f'(h)-f(0)]$$

So let's modify the Peano Kernel formula to look like the asymptotic one:

$$E_1(1,f) = \int_0^h f''(x)\left[\frac{-h^2}{12}\right]dx + \int_0^h f''(x)\left[\frac{x(x-h)}{2} + \frac{h^2}{12}\right]dx$$

$$= -\frac{h^2}{12}[f'(h)-f'(0)] + \int_0^h f''(x)\left[\frac{x^2}{2} - \frac{xh}{2} + \frac{h^2}{12}\right]dx$$

Now we integrate by parts

$$\int_0^h f''(x)\left[\frac{x^2}{2} - \frac{xh}{2} + \frac{h^2}{12}\right] dx =$$

$$\left[f''(x)\right]\left[\frac{x^3}{6} - \frac{x^2 h}{4} + \frac{h^2 x}{12}\right]\Bigg|_0^h -$$

$$\int_0^h f'''(x)\left[\frac{x^3}{6} - \frac{x^2 h}{4} + \frac{h^2 x}{12}\right] dx$$

We observe that

$$\frac{x^3}{6} - \frac{x^2 h}{4} + \frac{h^2 x}{12} = 0 \text{ at } x=0 \text{ and } x=h.$$

So we have only the last term.
Let's integrate by parts again!

$$-\int_0^h f'''(x)\left[\frac{x^3}{6} - \frac{x^2 h}{4} + \frac{h^2 x}{12}\right] dx = -f^{(4)}(x)\left[\frac{x^4}{24} - \frac{x^3 h}{12} + \frac{h^2 x^2}{24}\right]\Bigg|_{0=x}^h$$

$$+ \int_0^h f^{(4)}(x)\left[\frac{x^4}{24} - \frac{x^3 h}{12} + \frac{h^2 x^2}{24}\right] dx.$$

Now

$$\frac{x^4}{24} - \frac{x^3 h}{12} + \frac{h^2 x^2}{24} = \frac{1}{24}\left(x^4 - 2x^3 h + h^2 x^2\right)$$

$$= \frac{1}{24} x^2 \left(x^2 - 2xh + h^2\right)$$

$$= \frac{1}{24} x^2 (x-h)^2.$$

Clearly, this vanishes when $x=0$ and $x=h$, so we have

$$E(1,f) = -\frac{h^2}{12}\left[f'(h) - f'(0)\right] + \frac{1}{24}\int_0^h f^{(4)}(x)\, x^2 (x-h)^2\, dx$$

$$= -\left(\frac{1}{6}\right)\frac{h^2}{2!}\left[f'(h) - f(0)\right] + \frac{h^4}{4!}\int_0^h f^{(4)}(x)\,\frac{x^2(x-h)^2}{h^4}\, dx$$

Now

$$\frac{x^2(x-h)^2}{h^4} = \frac{x^2}{h^2}\cdot\frac{(x-h)^2}{h^2} = \left(\frac{x}{h}\right)^2 \cdot \left(\frac{x-h}{h}\right)^2$$

$$= \left(\frac{x}{h}\right)^2 \cdot \left(1 - \frac{x}{h}\right)^2$$

But $B_4(x) = x^2(x-1)^2$, so this is $B_4\left(\frac{x}{h}\right)$.

This establishes the $m=1$ case of the proof. To keep going, we observe

$$\frac{1}{24}\int_0^h x^2(x-h)^2\,dx = \frac{1}{24}\int_0^h x^2(x^2-2xh+h^2)\,dx$$

$$= \frac{1}{24}\int_0^h x^4-2x^3h+x^2h^2\,dx$$

$$= \frac{1}{24}\left[\frac{x^5}{5}-\frac{2x^4h}{4}+\frac{x^3h^2}{3}\right]\Bigg|_{x=0}^{h}$$

$$= \frac{1}{24}\left[\frac{h^5}{5}-\frac{2h^5}{4}+\frac{h^5}{3}\right]$$

$$= \frac{1}{24}\cdot\frac{\cancel{30}\,1}{30}\left[6h^5-15h^5+10h^5\right]$$

$$= \frac{h^5}{720}.$$

Now we do the same integration by parts trick to the term

$$\frac{1}{24}\int_0^h f^{(4)}(x)\,x^2(x-h)^2\,dx.$$

A skeptic would object that we have shown that a formula like the one we want holds for <u>some</u> numbers which are the integrals of <u>some</u> polynomials.

Why are these the <u>Bernoolli</u> numbers and polynomials?

Exercise (x/c)

Show that

$$B_j'(x) = j\,B_{j-1}(x) \qquad j \geq 4, \text{ even}$$

$$B_j'(x) = j\left[B_{j-1}(x) + B_{j-1}\right] \quad j \geq 3, \text{odd}$$

Tracking through the proof, these are exactly the debris from integration by parts. $\square$

Corollary. Suppose that the odd derivatives of f obey $f^{(2i-1)}(b) = f^{(2i-1)}(a)$ for all $i \in 1, \ldots, m$ (eg. when f is periodic with $b-a$ an integer multiple of the period) Then the trapezoid rule converges at rate h^{2m+2} for $\int_a^b f(x)\,dx$.

Proof. All of the lower powers of h drop out in the formula.

Corollary.

$$E(n,f) = \sum_{i=1}^{\infty} c_i h^{2i} \text{ for some } c_i.$$

(where $c_i = -\frac{B_{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right)$ do <u>not</u> depend on n).