

The Romberg Algorithm.

We have now given a recursive formula for the trapezoid rules with 2^n intervals:

$$R(n,0) = \frac{1}{2} R(n-1,0) + h \sum_{k=1}^{2^{n-1}} f[a + (2k-1)h]$$

where $h = \frac{b-a}{2^n}$.

~~We observe that we have~~

$$\int_a^b f(x) dx - R(n,0) = -\frac{1}{12} (b-a) h^2 f''(\xi)$$

~~but here ξ depends on h, a, b, n .~~

We have also proved (as a corollary of the Euler-Maclaurin formula) that

$$\int_a^b f(x) dx = R(n-1, 0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \dots$$

where the a_i depend on f but not h .
(Recall they are combinations of constants, Bernoulli numbers and odd derivatives of f evaluated at b and a .)

Idea: Let's use Richardson extrapolation to combine $R(n, 0)$, $R(n-1, 0)$ to kill the leading error term, and repeat to make it better.

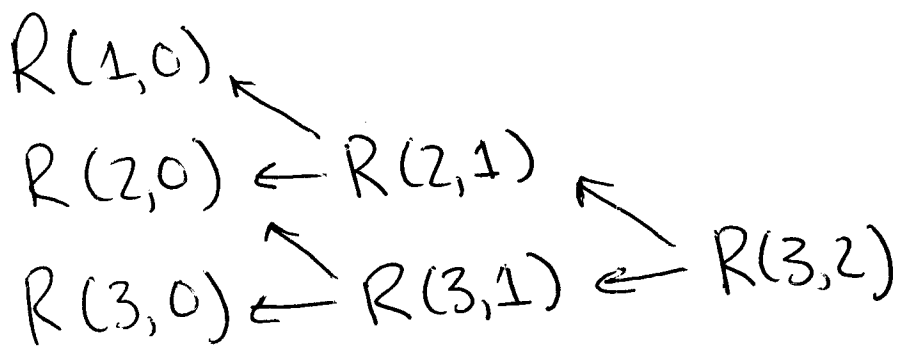
(2)

Recall the general form for Richardson extrapolation:

(3)

$$R(n, m) = \frac{4^m}{4^m - 1} R(n, m-1) - \frac{1}{4^m - 1} R(n-1, m-1)$$

which lets us compute



and the theorem

$$R(\overset{n}{\cancel{m}}, \overset{m}{\cancel{m}}) = \int_a^b f(x) dx + \sum_{k=m+1}^{\infty} A(k, m) \left(\frac{h}{2^n}\right)^{2k}$$

So the estimate $R(\overset{n}{\cancel{m}}, m)$ has error $O(h^{2m+2})$.

Of course, this should be tried out
on our favorite examples!

④

< Romberg-integration.nb >