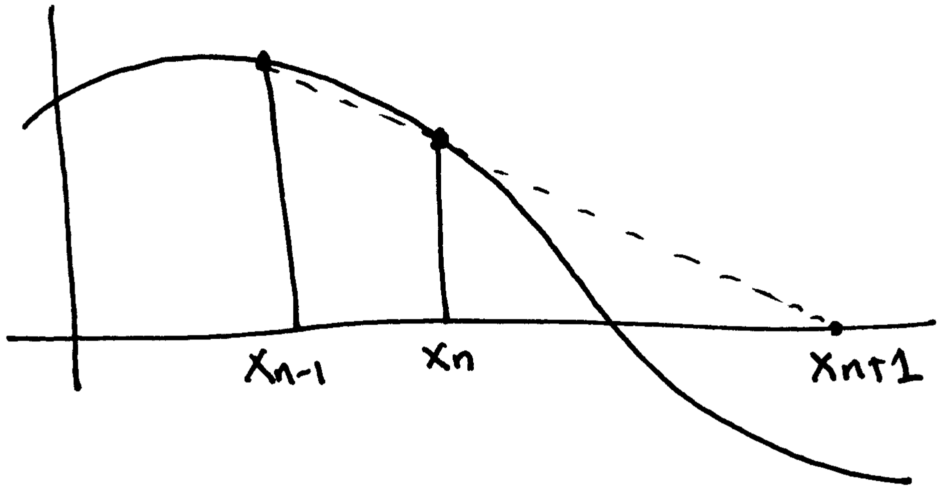
The secant method



We don't know $f'(x_n)$, but we can approximate it, we hope, with the slope

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

so we compute

$$\frac{-f(x_n)}{x_{n+1} - x_n} = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

or

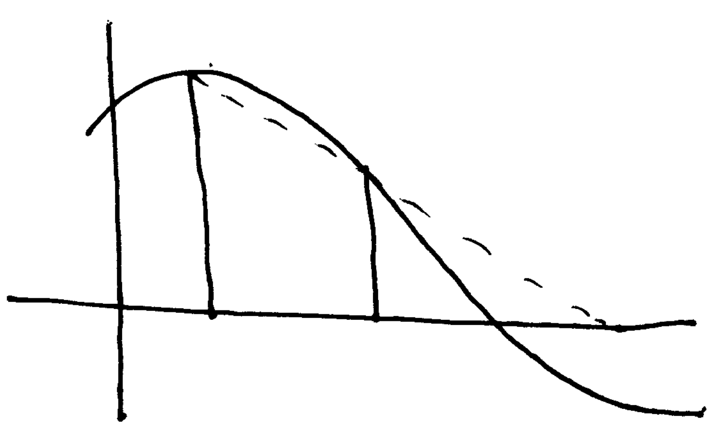
$$-f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = x_{n+1} - x_n$$

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

Keep in mind that this is going to get dicey as $f(x_n), f(x_{n-1})$ and x_n, x_{n-1} approach each other. So we should stop iterating when

$$\frac{|f(x_n) - f(x_{n-1})|}{|f(x_n)|} < \text{some relatively large number like } 10^{-6}$$

Convergence of the secant method.



It turns out to be the case that

$$e_{n+1} = -\frac{1}{2}\left(\frac{f''(\xi_n)}{f'(\gamma_n)}\right) e_n e_{n-1} \approx -\frac{1}{2}\left(\frac{f''(r)}{f'(r)}\right) e_n e_{n-1}$$

~~We see that~~ Suppose we can get

$$|e_{n+1}| \leq C|e_n|^\alpha$$

We want to solve for α, assuming there is a c with

$$|e_{n+1}| \leq c|e_n||e_{n-1}|.$$

If $c|e_0|, c|e_1| < D$, we see

$$c|e_2| \leq c|e_1|\, c|e_0| \leq D^2$$
$$c|e_3| \leq c|e_2|\, c|e_1| \leq D^2 D^1 = D^3$$
$$c|e_4| \leq c|e_3|\, c|e_2| \leq D^3 D^2 = D^5$$

In general,

$$c|e_n| \leq D^{\lambda_{n+1}}$$

where

$$\lambda_0 = 1, \quad \lambda_1 = 0, \qquad \lambda_n = \lambda_{n-1} + \lambda_{n-2}.$$

These numbers are the Fibonacci numbers (!).
In particular,

$$\lambda_n = \frac{1}{\sqrt{5}}\left(\gamma^n - \beta^n\right)$$

where $\gamma = \frac{1}{2}(1+\sqrt{5})$, $\beta = \frac{1}{2}(1-\sqrt{5})$.

~~To solve for α, we observe that in general,~~

We now know for <u>any</u> α.

$$
\begin{aligned}
|e_{n+1}| &\leq c\,|e_n|\,|e_{n-1}| \\
&= c\,|e_n|^{\alpha}\,|e_n|^{1-\alpha}\,|e_{n-1}| \\
&\approx c\,|e_n|^{\alpha}\left(c^{-1}D^{\lambda_{n+1}}\right)^{1-\alpha}\left(c^{-1}D^{\lambda_n}\right) \\
&= |e_n|^{\alpha}\,c^{1-(1-\alpha)-1}\,D^{(1-\alpha)\lambda_{n+1}+\lambda_n} \\
&= |e_n|^{\alpha}\,c^{\alpha-1}\,D^{-\alpha\lambda_{n+1}+\lambda_{n+1}+\lambda_n} \\
&= |e_n|^{\alpha}\,c^{\alpha-1}\,D^{\lambda_{n+2}-\alpha\lambda_{n+1}}.
\end{aligned}
$$

So we need to choose α so that $\lambda_{n+2}-\alpha\lambda_{n+1}\to 0$ as $n\to\infty$. Luckily, we know for $\alpha=\frac{1}{2}(1+\sqrt{5})$, we have

$$
\begin{aligned}
\lambda_{n+2}-\alpha\lambda_{n+1} &= \frac{1}{\sqrt{5}}\left(\alpha^{n+2}-\beta^{n+2}\right)-\frac{1}{\sqrt{5}}\left(\alpha^{n+2}-\alpha\beta^{n+1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\alpha\beta^{n+1}-\beta^{n+2}\right) \\
&= \frac{1}{\sqrt{5}}(\alpha-\beta)\beta^{n+1}
\end{aligned}
$$

where

$$\beta = \frac{1}{2}(1-\sqrt{5}) < 1,$$

so we see that $\alpha = \frac{1}{2}(1+\sqrt{5})$ works.

We comment that this means we have convergence faster than a linear method, but slower than a quadratic method.