Math 4510 - Lecture 1. O.D.E.

An ordinary differential equation is written in the form

$$x' = f(x,t), \quad \text{where } x = x(t)$$

or more generally

$$x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)}).$$

Picard-Lindelöf Theorem. If $f(x,t)$ is continuous in t and Lipschitz continuous in f, then the equation

$$x'(t) = f(x,t), \quad x(t_0) = x_0$$

has a unique solution on $t \in [x_0 - \epsilon, x_0 + \epsilon]$ for some $\epsilon > 0$.

Here a function $g(x)$ is <u>Lipschitz</u>

continuous if $\exists$ some C so that

$$|g(x) - g(y)| < C|x-y|$$

for all x, y.

Examples.

$$x' = Cx, \qquad x = e^{Ct+b}.$$

$$x'' = -x, \qquad x = \cancel{\cos t} \; \cancel{\text{or}} \; \cancel{\sin t}$$
$$= A\cos t + B\sin t.$$

$$x' = \frac{2x}{t}, \qquad x = At^2.$$

These are all called general or closed form solutions to the corresponding differential equations.

It is worth recalling that in general the solution to

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$$

will have n unknown constants.

These constants are determined by using the initial conditions. ~~For~~ ~~ex~~

Example.

$$x' = 2x, \quad x(0) = 1000.$$

The general solution is

$$x(t) = e^{2t+b}.$$

Plugging in

$$x(0) = e^{b} = 1000,$$

we see $b = \ln 1000$, and the specific solution is

$$x(t) = e^{2t + \ln 1000} = 1000e^{2t}.$$

Often these equations arise in applied problems. For instance, Example. The acceleration due to gravity of a falling object is crudely modelled by

$$x''(t) = -9.8$$

Compute the position $x(t)$ of a BASE jumper who dives from the tower of the Golden Gate bridge (227 m above San Francisco bay), at time $t=0$.

By integration, we see

$$x'(t) = -9.8t + C_1,$$
$$x(t) = -4.9t^2 + C_1 t + C_2.$$

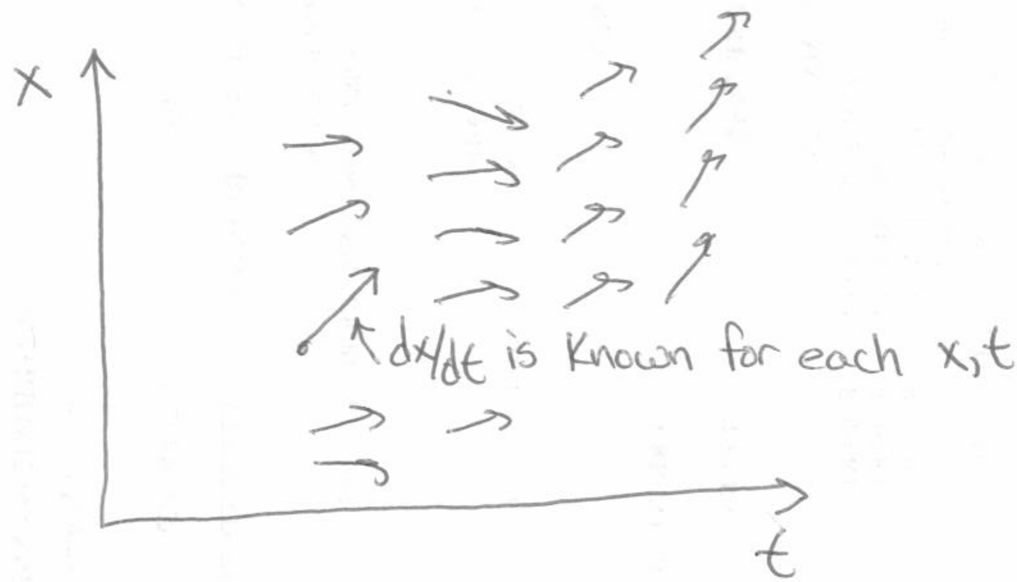
Plugging in

$$x(0) = 227. = C_2,$$
$$x'(0) = 0 = C_1,$$

we see

$$x(t) = -4.9t^2 + 227.$$

We can picture the solution of an ODE as a vector field



The solutions are curves tangent to the field at each point called integral curves.

VectorFieldsAndODES.nb.

~~Differential~~

Heat Transfer.nbp

Generally speaking, it is usually impossible to find closed form solutions to differential equations of interest in applications.

Example. On August 16, 1960 Cpt. Joseph Kittinger jumped from a helium balloon at 31,330 m. Suppose his parachute had failed (it didn't). Would he have been <u>accelerating</u> or <u>decelerating</u> when he hit the ground?

In general, falling objects in air obey

$$mv \frac{dv}{dz} = -mg + Kv^2$$

where K is a drag coefficient.

However, ~~a~~ the situation is complicated by the fact that K is proportional to air density and hence to air pressure (assuming constant temperature).

We end up with

$$mv\frac{dv}{dz} = -mg + K_0 v^2 e^{-z/\lambda}$$

where $\lambda = 7.4621 \times 10^3$ for Earth's atmosphere. For Kittinger's jump, we have the initial z already. We compute $K_0 \sim 0.21$.

Solving this equation for v is going to require more analytical trickery than we have on hand OR a good numerical method!

Taylor Series Methods.

Suppose

$$x(t+h) = x(t) + h x'(t) + \frac{h^2}{2} x''(t) + \dots$$

Now if

$$x' = f(x,t),$$

we can approximate

$$x(t+h) \approx x(t) + h\, f(x(t),t).$$

and use this to step forward in time.
This is called <u>Euler's method</u>.

The higher order Taylor methods are obtained by continuing this analysis.
For example,

Example. Consider the equation

$$x'(t) = 1 + x^2 + t^3.$$

We have

$$x''(t) = 2 x x' + 3t^2$$

$$x'''(t) = 2 x x'' + 2(x')^2 + 6t$$

$$x^{(4)}(t) = 2 x x''' + 2x'x'' + 4x'x'' + 6$$

$$= 2 x x''' + 6 x' x'' + 6.$$

We can now use

$$x(t+h) \approx x(t) + h x'(t) + \frac{h^2}{2} x''(t) + \frac{h^3}{6} x'''(t) + \frac{h^4}{4!} x^{(4)}(t)$$

to compute $x(t+h)$ recursively using the formulae above.

See Mathematica demo "Taylor Methods.nb"!