Boundary Value Problems 2

We want to solve

$$x''(t) = f(t, x, x')$$

$$a_0 x(a) - a_1 x'(a) = \gamma_1$$
$$b_0 x(b) + b_1 x'(b) = \gamma_2$$

We have observed that the solution to the initial value problem

$$x(a) = a_1 s - c_1 \gamma_1$$
$$x'(a) = a_0 s - c_0 \gamma_1$$

with $a_1 c_0 - a_0 c_1 = 1$, denoted $x(t;s)$, obeys our boundary conditions at a. At b, we define

$$\varphi(s) := b_0 x(b;s) - b_1 x(b;s) - \gamma_2 .$$

we are searching for s_* so that $\phi(s_*) = 0$. We want to compute

$$\phi'(s) = b_0 \frac{\partial}{\partial s} x(t;s) + b_1 \frac{\partial^2}{\partial s \partial t} x(t;s)$$

But how do we compute $\frac{\partial}{\partial s} x(t;s)$? Here's the idea: for any s we have

$$x''(t;s) = f(t, x(t;s), x'(t;s))$$

If we differentiate both sides w.r.t. s, we have an ode in the auxiliary function

$$y(t) = \frac{\partial}{\partial s} x(t;s)$$

given by

$$y''(t) = f_2(t, x(t;s), x'(t;s))\, y(t) + f_3(t, x(t;s), x'(t;s))\, y'(t).$$

with initial conditions

given by

$$y(a) = a_1, \quad y'(a) = a_0.$$

If we introduce the additional auxilary functions

$$z(t) = x'(t)$$
$$w(t) = y'(t)$$

this is a new system of ODE:

$$x'(t) = z(t)$$
$$z'(t) = f(t, x(t), z(t))$$
$$y'(t) = w(t)$$

$$w'(t) = f_2(t, x(t), z(t))\, y(t) + f_3(t, x(t), z(t))\, w(t)$$

where f_2 and f_3 are the partials of f w.r.t. the second and third variables.

Example.

$$x'' = -x + \frac{2(x')^2}{x} \qquad -1<t<1$$

with

$$x(1) = x(-1) = \frac{1}{e + 1/e}$$

The true solution is

$$x(t) = \frac{1}{e^t + e^{-t}} = \frac{1}{2}\operatorname{sech} x.$$

Since implementing the shooting method is part of the project, we won't do this in Mathematica.

But we will compute the system to solve:

$$x'(t) = z(t)$$

$$z'(t) = -x(t) + \frac{2(x')^2}{x}$$

$$y'(t) = w(t)$$

$$w'(t) = \left(-1 - \frac{2(\cancel{x'} z)^2}{x^2}\right) y(t) + \frac{4 \cancel{x}\, z(t)^2}{x(t)} w(t)$$

with initial conditions

$$x(-1) = \frac{1}{e + 1/e}$$

$$z(-1) = 5$$

$$y(-1) = 0$$

$$w(-1) = 1$$

We would solve this and use

$$\varphi'(s) = y(1), \quad \varphi(s) = x(1) - \frac{1}{e + 1/e}.$$

as ~~input~~ data for Newton's method.

~~Note: there are various pr~~

Remark. We replicated a general theorem about how solutions of an ODE depend on parameters in the equation. The general theorem here is due to Peano, and can be found on p 95 of Hartmann's O.D.E. book.

What can go wrong here?

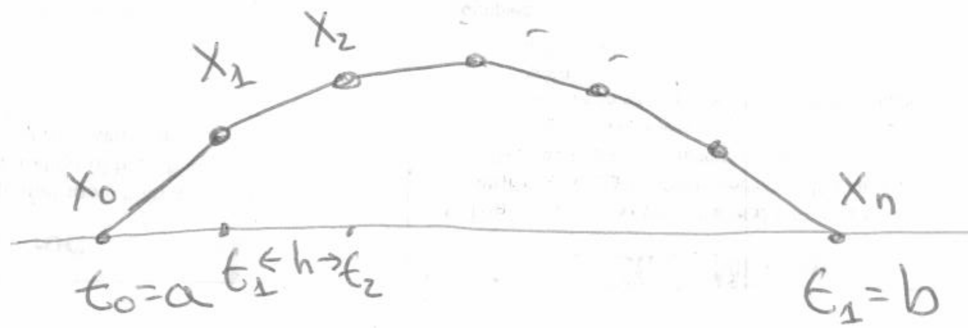
* How do we guess s_0?
* Does Newton's method converge?
* What if the problem is very sensitive to s? This is <u>likely</u>, as

$$|x(b;s) - x(b,s+h)| \approx e^{b-a} h$$

* How many times must we solve the IVP? With what stepsizes?

For all these reasons, it can be valuable to have different methods where the convergence to a solution is better controlled.

Idea. Suppose we choose to divide the interval $[a,b]$ into equal intervals by $t_0, t_1, \ldots, t_n$.



We think of the set of

$$x_i = x(t_i)$$

as a set of $n+1$ variables, obeying some equations, and solve the equations.

Suppose that we are in the linear case

$$x''(t) = u(t) + v(t)\,x(t) + w(t)\,x'(t)$$

If we let

$$x'(t) \approx \frac{1}{2h}\left[x(t+h) - x(t-h)\right]$$

$$x''(t) \approx \frac{1}{h^2}\left[x(t+h) - 2x(t) + x(t-h)\right]$$

We can write the ODE as a system

$$\frac{1}{h^2}\left[x_{i+1} - 2x_i + x_{i-1}\right] = u(t_i) + v(t_i)x_i + w(t_i)\left[\frac{1}{2h}\left[x_{i+1} - x_{i-1}\right]\right].$$

of linear equations in the x_i, which can be written

$$\left(\frac{1}{h^2} + w_i\frac{1}{2h}\right)x_{i-1} + \left(-\frac{2}{h^2} - v_i\right)x_i + \left(\frac{1}{h^2} - \frac{w_i}{2h}\right)x_{i+1} = u_i$$

Multiplying through by $-h^2$, we get

$$-\left(1 + hw_i/2\right)x_{i-1} + \left(2 + h^2 v_i\right)x_i + \left(\frac{h}{2}w_i - 1\right)x_{i+1} = -h^2 u_i$$

Now the free variables are really only $x_1, \ldots, x_{n-1}$ since the boundary conditions specify $x_0 = x(a)$ and $x_n = x(b)$. We have a matrix in the form

$$\begin{bmatrix} \bullet & \bullet & & & & \\ \bullet & \bullet & \bullet & & 0 & \\ & \bullet & \bullet & \bullet & & \\ & & \bullet & \bullet & \bullet & \\ & 0 & & \ddots & \ddots & \ddots \\ & & & & \bullet & \bullet \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} \uparrow \\ -h^2 v_i \\ \downarrow \end{bmatrix}$$

This is a special type of matrix, called Tridiagonal, for which $A\vec{x} = \vec{b}$ can be solved in time $O(n)$, as we will see in the next unit!

This is called a <u>discretization</u> or a <u>finite</u> <u>element</u> method.

(Mathematica demo)