Special Matrices

Symmetric positive definite matrices satisfy $A = A^T$, $x^T A x > 0$ for $x \neq 0$.
We Know

Proposition. X nonsingular $\Rightarrow$ A s.p.d $\Leftrightarrow$ $X^T A X$ spd

If A is spd and H is a principal submatrix then H is spd.

A is spd $\Leftrightarrow$ $A = A^T$, all eigenvalues positive.

A is spd $\Rightarrow$ $a_{ii} > 0$, $\max_{ij} |a_{ij}| = \max_i a_{ii} > 0$.

A is spd $\Leftrightarrow$ $\exists$ a unique lower triangular L with positive diagonal entries so $A = LL^T$. This is called the <u>Cholesky factorization</u> of A.

Proof. (Of last one) If $A = LL^T$ for any nonsingular L then $A^T = (LL^T)^T = LL^T = A$, Now by part 1, $\begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix}$ is spd

so by part 2, the pricipal submatrix $\tilde{A}_{22}$ is spd. So $\tilde{A}_{22} = \tilde{L}\tilde{L}^T$ and we can complete from there. $\square$

This leads to Cholesky algorithm.

Sparse matrices.

$$A = \begin{bmatrix} 1 & & & & -1 \\ & \ddots & & & \cdot 1 \\ & & \ddots & & \vdots \\ & & & \ddots & 1 \\ \cdot 1 & \cdot 1 & \cdots & & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \cdot 1 & \cdot 1 & \cdots & \cdot 1 \\ -1 & 1 & & & \\ \cdot 1 & & & & 1 \end{bmatrix}$$

and

$$x^T A x = x^T L L^T x = (L^T x)^T (L^T x) > 0$$

for $x \neq 0$. This proves $\Leftarrow$.

So suppose A spd. We prove $A = LL^T$ be induction on n. Consider

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \frac{A_{12}^T}{\sqrt{a_{11}}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \frac{A_{12}}{\sqrt{a_{11}}} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & A_{12} \\ A_{12}^T & \tilde{A}_{22} + \frac{A_{12}^T A_{12}}{a_{11}} \end{bmatrix}$$

so

$$A_{22} = \tilde{A}_{22} + \frac{A_{12}^T A_{12}}{a_{11}}$$

is symmetric.