# The Variational Approach and Elliptic PDE ①

We want to show that harmonic functions obey a really amazing property:

$\Delta u = 0$ on $\Omega$ with $u = f$ on $\partial\Omega$ $\iff$ $u$ is the minimizer of the functional $D(v,v) = \int_\Omega \nabla v \cdot \nabla v \, dVol$ among functions with $u = f$ on $\partial\Omega$.

This will lead us to some new numerical methods for the Laplace, and in general for elliptic PDE problems.

Green's Identity 1. For functions $u$ and $v$ on a domain $\Omega$ with smooth boundary, $S$

$$\int_S (v \nabla u) \cdot \vec{n} \, dArea = \int_\Omega v \Delta u + \nabla u \cdot \nabla v \, dVol.$$

We compute

$$\nabla \cdot (v \nabla u) = \sum_i \frac{\partial}{\partial x_i} (v \nabla u)_i$$

$$= \sum_i \frac{\partial}{\partial x_i} v \frac{\partial u}{\partial x_i}$$

$$= \sum_i \left( \frac{\partial v}{\partial x_i} \right) \cdot \left( \frac{\partial u}{\partial x_i} \right) + v \frac{\partial^2 u}{\partial x_i^2}$$

$$= \nabla v \cdot \nabla u + v \Delta u.$$

Now this is just the divergence theorem.

Green's Identity 2. If $\Omega$ is a domain with smooth boundary $S$, then

$$\int_S v(\nabla u \cdot \vec{n}) - u(\nabla v \cdot \vec{n}) \, dArea =$$

$$= \int_\Omega (v\Delta u - u\Delta v) \, dVol$$

(Just switch $u$ and $v$ in identity 1 and subtract.)

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Now suppose $u, v$ both equal $f$ on $\partial\Omega$, $\Delta u = 0$.
We have

$$D(u-v, u+v) = \int_\Omega (\nabla u - \nabla v) \cdot (\nabla u + \nabla v) \, dVol$$

$$= D(u,u) - D(v,v)$$

But

$$\int_\Omega \nabla(u-v) \cdot \nabla(u+v) \, dVol = \quad \text{Green's Id 1.}$$

$$\cancel{\int_S (u-v)(\nabla(u+v)\cdot\vec{n}) \, dArea}$$

$$- \int_\Omega (u-v) \Delta(u+v) \, dVol$$

$$= - \int_\Omega (u-v) \Delta v \, dVol \quad \text{since } \Delta u = 0$$

$$= \int_\Omega (u-v) \Delta(u-v) \, dVol$$

Green's Id 1.

$$= - \int_\Omega \nabla(u-v) \cdot \nabla(u-v) \, dVol + \cancel{\int_S (u-v) \nabla(u-v) \vec{n} \, dA}$$

$$\leq 0.$$

This proves that for any $v$ on $\Omega$ obeying the same boundary conditions,

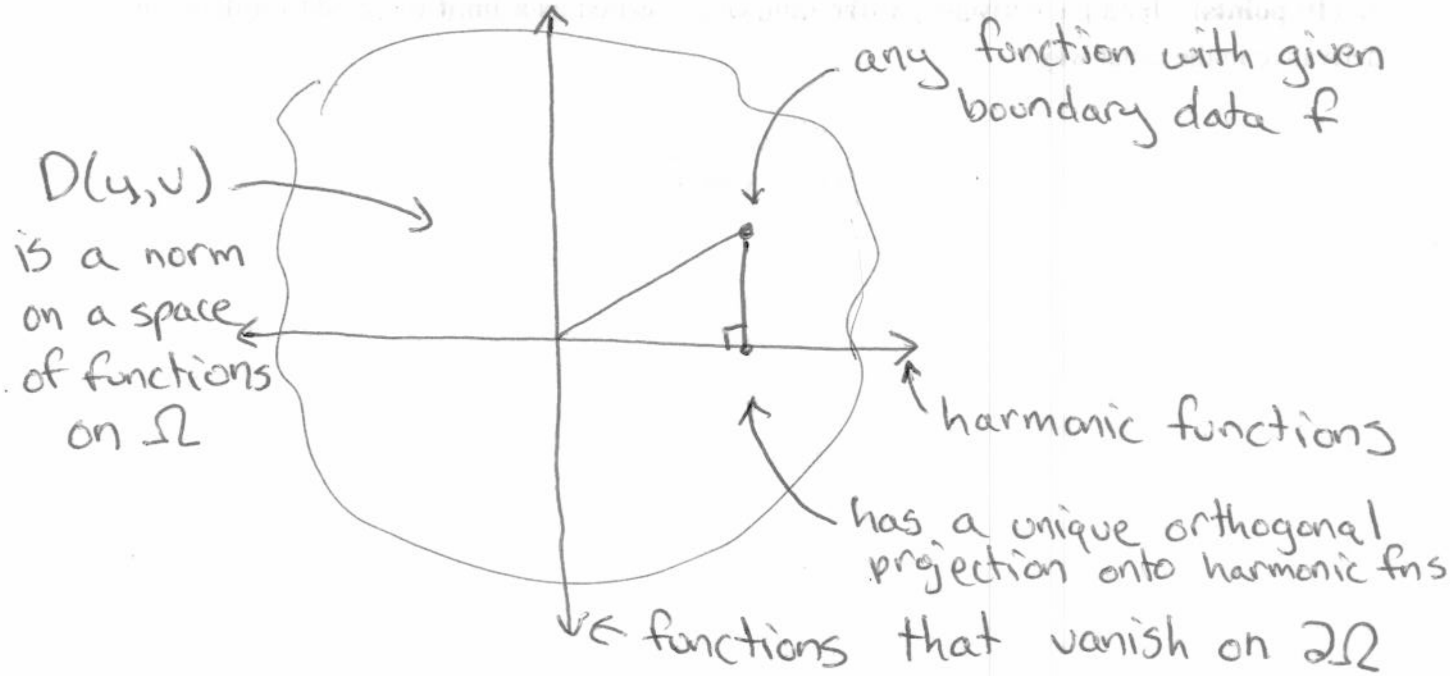
$$D(u,u) \leq D(v,v)$$

if $\Delta u = 0$ on $\Omega$.

Remarks.

1) This does not prove that such a $u$ exists, but we could do this properly (eg 2F of Folland) and prove that as well.
2) We can use this to obtain specific solutions in many cases.

3) There's a pretty picture here:



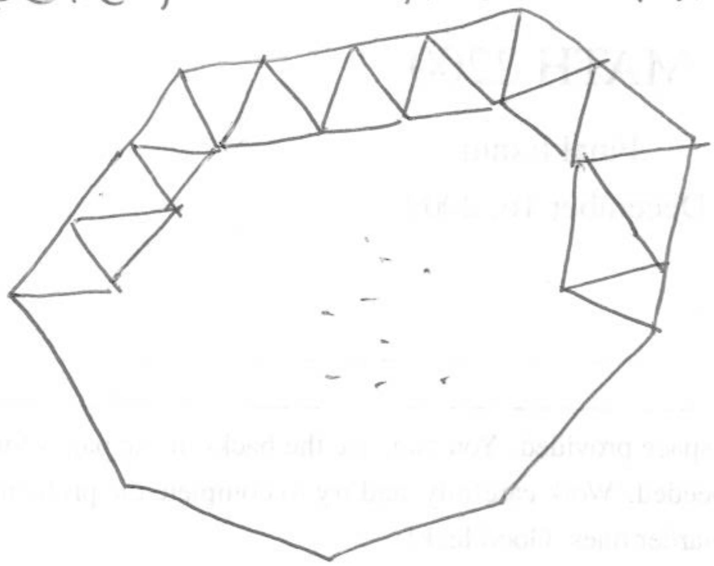

4) Actually, we could do this for a general elliptic problem (but the details of D would change, cf chapter 7 of Folland).

5) We need one more case:

$\Delta u = r \quad \Leftrightarrow \quad u$ mimimizes

$$D(u,u) = \int_\Omega \nabla u \cdot \nabla u + 2ru \, dVol$$

A (better) method of finite elements. 



We now use this observation to build a better solver.

Idea: Suppose we triangulate $\Omega$ and allow values at mesh points to vary, assuming $u$ is piecewise linear on triangles. We could take

mesh point $(x_i, y_i, z_i)$

$\longleftrightarrow$ now try to write down $\Delta u$ for vertex in terms of mesh vals.

(old approach)

(new idea)



$\longleftrightarrow$ write down

$$\int_{\Delta_i} \nabla u \cdot \nabla u \; dVol$$

as a linear function of the ~~ne~~ vertices of the triangle and minimize the function

$$\sum_i \int_{\Delta_i} \nabla u \cdot \nabla u \, dVol$$

if $u$ is linear on this triangle $\nabla u$ is constant

This is (a) finite element method and we will see it's pretty powerful.