Iterative Refinement

What if a numerical linear algebra solution to $Ax = b$ is not accurate enough for our application?

As usual, let

$$r = Ax_i - b$$

be the residual. Now solve

$$Ad = r$$

for d and let

$$x_{i+1} = x_i - d.$$

Repeat.

At first glance, this looks rather

puzzling. If we can solve $Ad = r$ accurately, then

$$\begin{aligned} Ax_{i+1} &= Ax_i - Ad \\ &= r + b - r \\ &= b \end{aligned}$$

as expected, but if we could do that, why couldn't we solve $Ax = b$ in the first place?

If we can't solve either problem accurately, why should this help?

Theorem. Suppose r is computed in double (your usual) precision and $\kappa(A)\epsilon < \frac{1}{3n^3 g+1}$ where g is the pivot growth factor and n the dimension of A. Then repeated iterative refinement converges with

our solution   true solution

$$\frac{\| x_i - A^{-1}b \|_\infty}{\| A^{-1} b\|_\infty} = O(\epsilon)$$

true solution

Proof sketch. We will show that (taking only leading error terms into account).

$$\| x_{i+1} - x \|_\infty \le \frac{\kappa(A)\,\varepsilon}{c} \| x_i - x \|_\infty$$

Now if we do the matrix multiply and subtraction in double our usual precision,

$$\begin{aligned} r &\approx Ax_i - b + \epsilon |Ax_i - b| \\ &\approx Ax_i - b + f \end{aligned}$$

Now when we solve $Ad=r$, we get

$$(A+\delta A)\,d = r$$

where

$$\|\delta A\|_\infty \le 3n^3 g\cdot\epsilon\cdot\|A\|_\infty$$

from before. Now we assume $x_{i+1}=x_i-d$ and ignore roundoff here (because we are working in twice our usual precision). So

$$d=(A+\delta A)^{-1}r$$

~~Now it turns out that~~

$$\cancel{(A+\epsilon X)^{-1}=A^{-1}-\epsilon A^{-1}XA^{-1}+O(\epsilon^2)}$$

~~so~~

$$\cancel{d\approx(A^{-1}-A^{-1}\delta A^{-1}A^{-1})r}$$

$$\cancel{\approx(I-A^{-1}\delta A^{-1})A^{-1}r}$$

or

$$d = \left(A \left(I + A^{-1} \delta A \right) \right)^{-1} r$$

$$= \left(I + A^{-1} \delta A \right)^{-1} A^{-1} r$$

$$= \left(I + A^{-1} \delta A \right)^{-1} A^{-1} \left(A x_i - b + f \right)$$

$$= \left(I + A^{-1} \delta A \right)^{-1} \left(x_i - A^{-1} b + A^{-1} f \right)$$

Now $A^{-1} b = x$. Further we know from before that in general since $\| - A^{-1} \delta A \| < 1$, we have

$$I - \left(- A^{-1} \delta A \right) = \sum_{i=0}^{\infty} \left(- A^{-1} \delta A \right)^i$$

$$= I - A^{-1} \delta A + \left(A^{-1} \delta A \right)^2 + \dots$$

Rounding off higher order terms,

$$\approx \left(I - A^{-1} \delta A \right)$$

so

$$d \approx (I - A^{-1}\delta A)(x_i - x + A^{-1}f)$$

$$\approx x_i - x + A^{-1}f - A^{-1}\delta A(x_i - x)$$

missing term is small

so

$$x_{i+1} - x = (x_i - d) - x$$

$$= (x_i - x) - d$$

$$= A^{-1}\delta A(x_i - x) - A^{-1}f.$$

Now we take norms on both sides and estimate

$$\| x_{i+1} - x \| \leq \|A^{-1}\| \, \|\delta A\| \, \|x_i - x\| + \|A^{-1}\| \epsilon \|Ax_i - b\|$$

$$\leq \|A^{-1}\| \, \|\delta A\| \, \|x_i - x\| + \|A^{-1}\| \epsilon \|A\| \, \|x_i - x\|$$

$$\leq \left(\|A^{-1}\| (3n^3 g) \epsilon \|A\| + \|A^{-1}\| \|A\|\right) \|x_i - x\|$$

$$\leq \|A^{-1}\| \, \|A\| (3n^3 g + 1) \epsilon \; \|x_i - x\|$$

$$\leq \kappa(A) \epsilon / 3n^3 g+1 \; \|x_i - x\|. \quad \square$$