Runge-Kutta Methods

The basic idea of an RK method is to solve an ODE without requiring you to differentiate $f(x,t)$ in

$$x' = f(x,t).$$

We will need Taylor's theorem in two variables. The statement turns out to be

$$f(x+h, y+k) = \sum_{i=0}^{\infty} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x,y)$$

where

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 = h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\partial y} + k^2\frac{\partial^2}{\partial y^2}$$

(and so forth). For example,

$$f(x+h, y+k) = f(x,y) + hf_x + kf_y + \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right) + \ldots$$

As before, there's an error term if we truncate the series:

$$f(x+h, y+k) = \sum_{i=0}^{n-1} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x,y) + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(\bar{x},\bar{y})$$

where $(\bar{x},\bar{y})$ is a point on the line segment between (x,y) and $(x+h, y+k)$.

Basic Framework of RK2:

Suppose we are willing to take two function evaluations,

$$K_1 = hf(t,x)$$
$$K_2 = hf(t+\alpha h, x+\beta K_1)$$

and assume

$$x(t+h) \approx x(t) + \omega_1 K_1 + \omega_2 K_2$$

We want to choose $\omega_1, \omega_2, \alpha, \beta$ so that ~~[illegible]~~ our formula reproduces as many terms as possible in

$$X(t+h) = x(t) + h x'(t) + \frac{h^2}{2} x''(t) + \dots$$

Now our formula can be expanded as

$$x(t) + \omega_1 h f(\overset{t,x}{\cancel{x,t}}) + \omega_2 h f(t+\alpha h, x+\beta h f(x,t)).$$

(Applying Taylor's theorem to the second term)
(and letting $f(t,x) = f$)

$$f(t+\alpha h, x+\beta h f(\cancel{x,t})) = f + \alpha h f_t + \beta h f f_x + \frac{1}{2}\left(\alpha h \frac{\partial}{\partial t} + \beta h f \frac{\partial}{\partial x}\right)^2 f(\bar{x}, \bar{y})$$

We can then expand our original formula as:

$$x(t) + (\omega_1 + \omega_2) h f + \alpha h^2 \omega_2 f_t + \beta h^2 f \omega_2 f_x + O(h^3)$$

Now let's return to

$$x(t+h) = x(t) + h x'(t) + \frac{h^2}{2} x''(t) + O(h^3)$$

we know

$$x'(t) = f(t,x)$$

$$x''(t) = f_t + f_x x'(t) = f_t + f_x f$$

applying these, we have

$$x(t+h) = x(t) + hf + \frac{h^2}{2} f_t + \frac{h^2}{2} f_x f + O(h^3)$$

$$\approx x(t) + (\omega_1 + \omega_2) hf + \alpha \omega_2 h^2 f_t + \beta \omega_2 h^2 f_x f + O(h^3)$$

This gives us a set of equations:

$$\begin{aligned} \omega_1 + \omega_2 &= 1 \\ \alpha \omega_2 &= 1/2 \\ \beta \omega_2 &= 1/2 \end{aligned} \quad \underset{\text{e.g.}}{\Longrightarrow} \quad \begin{aligned} \omega_1 &= 1/2 = \omega_2 \\ \alpha &= 1 \\ \beta &= 1 \end{aligned}$$

This gives us the formula:

$$x(t+h) = x(t) + \frac{h}{2} f(t,x) + \frac{h}{2} f(t+h, x+hf(t,x))$$

~~Now there~~
which is called the Runge-Kutta formula of order 2.

There are certainly higher-order Runge-Kutta formulae. For example:

4th order RK formula:

$$x(t+h) = x(t) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = hf(t, x)$$

$$K_2 = hf(t + \tfrac{1}{2}h, x + \tfrac{1}{2}K_1)$$

$$K_3 = hf(t + \tfrac{1}{2}h, x + \tfrac{1}{2}K_2)$$

$$K_4 = hf(t + h, x + K_3)$$

We now try these out on some example problems to get a feel for the method.