Bayes' Theorem (2).

With the "Wolf Blitzer" example, we practiced splitting the sample space into 2 parts (human, lycanthrope).

We can split into any number of parts and do a similar trick.

Theorem. If $P(B) > 0$ and if $A_1, A_2, \ldots$ form a partition of the sample space, with all $P(A_j) > 0$, then

$$P(A_k \mid B) = \frac{P(A_k)\,P(B \mid A_k)}{\sum_j P(A_j)\,P(B \mid A_j)}$$

Example. In your ~~certain~~ dorm, there are 5 floors, each with an equal number of students. The percentage of engineering majors on each floor is 80%, 52%, 74%, 67% and 29%.

You meet an engineering major from your dorm. Find the probability they live on floor 4.

Let $A_j = \{\text{the student lives on floor } j\}$. Then $P(A_j) = 1/5$. Let $B = \{\text{engineering major}\}$.

We know

$$P(B|A_1) = 0.8, \ \ldots, \ P(B|A_5) = 0.29$$

We want $P(A_4|B)$.

$$P(A_4|B) = \frac{P(A_4)P(B|A_4)}{P(A_1)P(B|A_1) + \cdots + P(A_5)P(B|A_5)}$$

$$= \frac{0.20 \times 0.67}{0.2\times0.8 + 0.2\times.52 + 0.2\times0.74 + 0.2\times0.67 + 0.2\times0.29}$$

$$= 0.22184$$

Interpretation. The overall $P(B) = 0.604$ (the denominator above). Since the 4th floor has 67% engineers, it has more engineers (proportional to # of students) than average - meaning that a randomly selected engineer is more likely than average to come from floor 4.

Example. (Double flips)

A coin game goes as follows. We flip a fair coin until we get the first head (suppose this takes K flips).

~~Then~~

This means that we flipped $\underbrace{T, T, \ldots T, H}_{K \text{ times}}$.

We then flip K coins again. We win if all K are heads.

a) What is $P(\text{winning})$?

b) Given that you won, what is the probability that K was 4?