Differential Forms

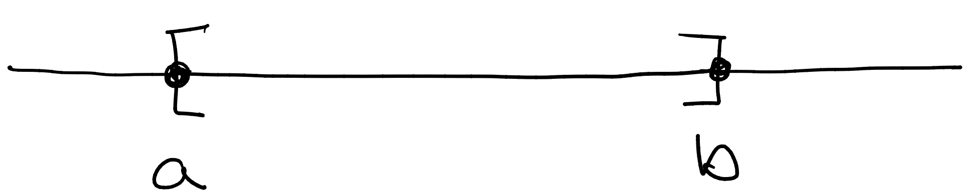
8.1/8.2 Differential Forms.

We now understand how to differentiate and integrate in multivariable calculus.

We want to connect the two operations by proving the fundamental theorem of calculus. In one variable, this is

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

whenever $f(x) = F'(x)$. Now



So we can observe: (in our new language)

$[a,b]$ is a region Ω

$[a,b]$ is the closure of (a,b)

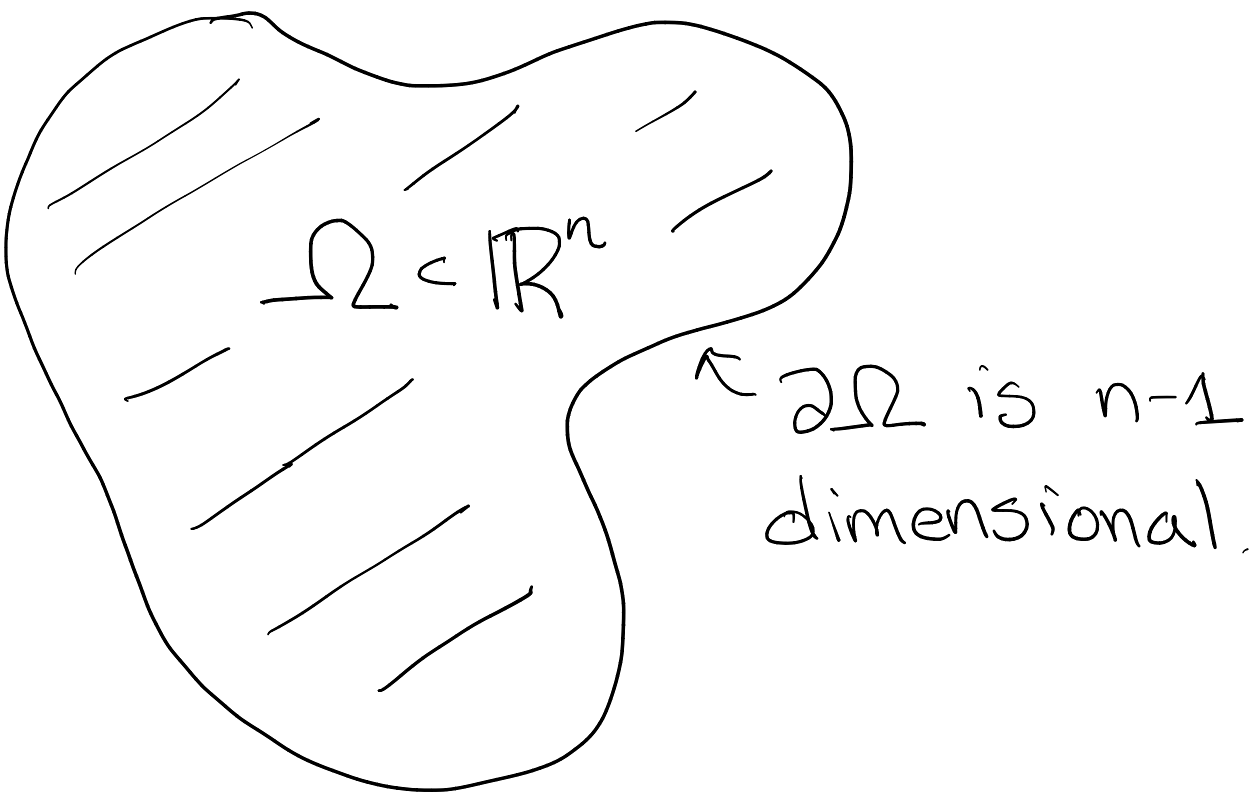
$\{a,b\} = \mathrm{bdy}(\Omega) = \partial\Omega$ is the set of frontier or boundary points of Ω.

So the theorem has the structure

$$\int_{\Omega} dF(x)\,dx = \int_{\{b\}} F(x)\,dx - \int_{\{a\}} F(x)\,dx$$

if we realize that "0-dimensional integration" is just evaluation at the point.

To extend this to higher dimensional regions, we'll need to keep track of several things:



F something that can be integrated over $(n-1)$-dimensional manifolds

dF something that can be integrated over n-dimensional manifold and our goal will be to prove Stokes's Theorem:

$$\int_{\Omega} dF = \int_{\partial\Omega} F$$

So what <u>are</u> F and dF?

They are a special kind of vector-valued function called a <u>differential</u> <u>form</u>.

Defining differential forms will take a few classes.

Definition. The vectorspace of linear maps $\mathbb{R}^n \to \mathbb{R}$ is called the <u>dual space</u> $(\mathbb{R}^n)^*$.

The dual space is an n-dimensional vector space. If $\mathbb{R}^n$ has the <u>standard basis</u> $\vec{e}_1, \ldots, \vec{e}_n$ then $(\mathbb{R}^n)$ has a <u>standard dual basis</u> $dx_1, \ldots, dx_n$ defined by

$$dx_i(\vec{e}_j) = \delta_{ij}$$

so

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow dx_i(\vec{v}) = v_i$$

Homework: Prove that $\{dx_i\}$ is a basis for $(\mathbb{R}^n)^*$.

Now we can extend this idea:

Definition. The vector space of alternating multilinear functions $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}} \to \mathbb{R}$ is denoted $\Lambda^k(\mathbb{R}^n)$.

We <u>define</u> $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$.

functions with no inputs are constant

We know that

$\Lambda^n(\mathbb{R}^n)$ is one-dim. (det)

We call this space "alt-n-k" or "Lambda-n-k".

and

$$\Lambda^1(\mathbb{R}^n) = (\mathbb{R}^n)^* = n\text{-dimensional}$$

Now we want to build a basis for the "middle guys" $\Lambda^k(\mathbb{R}^n)$.

Definition. A k-index I in $\mathbb{R}^n$ is an ordered $(i_1, \dots, i_k)$ with each $i_j \in \{1, \dots, n\}$.

Note: When we say "ordered", we mean that the order of the i_j matters: $(7,3,7)$ and $(3,7,7)$ are <u>different</u> 3 indices in $\mathbb{R}$. There's also no requirement that the indices i_j are all different.

Given a k-index I in $\mathbb{R}^n$, we define

$$dx_I(\vec{v}_1, \dots, \vec{v}_k) = \det V_I$$

$$= \det \begin{bmatrix} dx_{i_1}(\vec{v}_1) & \cdots & dx_{i_1}(\vec{v}_k) \\ \vdots & \ddots & \vdots \\ dx_{i_k}(\vec{v}_1) & \cdots & dx_{i_k}(\vec{v}_k) \end{bmatrix}$$

where V is the $n \times k$ matrix $V = \begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_k \\ \downarrow & & \downarrow \end{bmatrix}$ and V_I is the submatrix formed by taking rows $i_1, \dots, i_k$.

Example. $n=3$, $k=2$.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

Then if $I=(1,3)$,

$$V_I = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \qquad dx_I(\vec{v}_1, \vec{v}_2) = 11$$

and if $I=(2,2)$

$$V_I = \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix} \qquad dx_I(\vec{v}_1, \vec{v}_2) = 0.$$

Proposition. The dimension of $\Lambda^k(\mathbb{R}^n)$ is $\binom{n}{k}$ and a basis for $\Lambda^k(\mathbb{R}^n)$ is given by $\{dx_I\}$ where $I=(i_1,\dots,i_k)$ with $i_1 < i_2 < \dots < i_k$.

Proof. Homework.

Example. $\Lambda^2(\mathbb{R}^3)$ has dimension

$$\binom{3}{2} = \frac{3!}{2!\,(3-2)!} = \frac{6}{2} = 3.$$

A basis for $\Lambda^2(\mathbb{R}^3)$ is given by

$$dx_{12},\ dx_{13},\ dx_{23}$$

Definition. The <u>wedge product</u>

$\wedge : \Lambda^k(\mathbb{R}^n) \times \Lambda^\ell(\mathbb{R}^n) \to \Lambda^{k+\ell}(\mathbb{R}^n)$

is the unique bilinear map which has

$$dx_I \wedge dx_J = dx_{(I,J)}$$

where if $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_\ell)$ then

$$(I,J) = (i_1, \dots, i_k, j_1, \dots, j_\ell)$$

is the concatenation of I and J.

Examples.

$$dx_{12} \wedge dx_3 = dx_{123}$$

$$dx_{17} \wedge dx_{25} = dx_{1725}$$

Important example. Suppose

$$\omega = a_1 dx_1 + a_2 dx_2 \in \Lambda^1(\mathbb{R}^2)$$

$$\eta = b_1 dx_1 + b_2 dx_2 \in \Lambda^1(\mathbb{R}^2).$$

Then we compute (bilinearity)

$$\begin{aligned} \omega \wedge \eta &= a_1 b_1 dx_1 \wedge dx_1 \\ &\quad + a_1 b_2 dx_1 \wedge dx_2 \\ &\quad + a_2 b_1 dx_2 \wedge dx_1 \\ &\quad + a_2 b_2 dx_2 \wedge dx_2 \\ &= a_1 b_1 dx_{11} + a_1 b_2 dx_{12} \\ &\quad + a_2 b_1 dx_{21} + a_2 b_2 dx_{22} \end{aligned}$$

Now we stop to think. Exchanging entries in a multi-index exchanges rows in a determinant. So it changes the sign of the result (alternating). This means

$$dx_{\underline{11}} = -dx_{\underline{11}} = 0$$

$$dx_{\underline{21}} = -dx_{\underline{12}}$$

$$dx_{\underline{22}} = -dx_{\underline{22}} = 0$$

so

$$\omega \wedge \eta = (a_1 b_2 - a_2 b_1)\, dx_{\underline{12}}$$

Proposition. The wedge product
$\wedge : \Lambda^k(\mathbb{R}^n) \times \Lambda^\ell(\mathbb{R}^n) \to \Lambda^{k+\ell}(\mathbb{R}^n)$
is associative and obeys

$$\omega \wedge \eta = (-1)^{k\ell}\, \eta \wedge \omega.$$

Proof. Associativity for basis elements is formal:

$$(dx_I \wedge dx_J) \wedge dx_K = dx_{((I,J),K)}$$

but $((I,J),K)$ is the concatenation of the multi-indices I, J, K, so it is the same as $(I,(J,K))$.

Thus

$$(dx_I \wedge dx_J) \wedge dx_K = dx_I \wedge (dx_J \wedge dx_K)$$

Now we've already observed that if I' is related to I by swapping a pair of indices,

$$dx_{I'} = -dx_I.$$

So

$$dx_I \wedge dx_J = dx_{IJ}$$
$$= \pm dx_{JI} = dx_J \wedge dx_I$$

where the sign is determined by how many swaps it takes to rearrange

$$i_1 \dots i_k j_1 \dots j_\ell \to j_1 \dots j_\ell i_1, \dots i_k$$

but it takes k swaps to move j_1 to the front

$$i_1 \cdots i_k j_1 \cdots j_\ell \to j_1 i_1 \cdots i_k j_2 \cdots j_\ell$$

and then k more for j_2, and eventually $k\ell$ total swaps.

Thus

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega. \quad \square$$

This allows us to prove

Lemma. If $I = (i_1, \dots, i_k)$ then

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Note the <u>lack</u> of parentheses!

Let's take stock. We've defined vector spaces $\Lambda^k(\mathbb{R}^n)$. We are now going to consider vector valued functions. whose values lie in these spaces.

Definition. If $U \subset \mathbb{R}^n$ is an open set, and $\omega : U \subset \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$ is a smooth function, we say ω is a <u>differential</u> <u>k-form</u> on U.

The set of k-forms $A^k(U)$ is an (infinite-dimensional) vector space, which just means that we may add and scalar multiply k-forms using the vector space structure on $\Lambda^k(\mathbb{R}^n)$.

Since our basis for $\Lambda^k(\mathbb{R}^n)$ doesn't have a natural order, it's easier to think of "basis functions" for a form instead of "coordinate functions", and write

$$\omega = \sum_{\substack{I \text{ an} \\ \text{increasing} \\ k\text{-index} \\ \text{in } \mathbb{R}^n}} f_I \, dx_I$$

where $f_I : U \subset \mathbb{R}^n \to \mathbb{R}$.

Now we're going to write

$$\omega = \sum_I{}' f_I \, dx_I \in A^k(U)$$

for an open $U \subset \mathbb{R}^n$.

On the right hand side, we know the $\{dx_I\}$ span $\Lambda^k(\mathbb{R}^n)$ so every vector in this vector space can be written as a linear combination of dx_I's.

The f_I's are the coefficients.

Actually, a subset of $\{dx_I\}$ (the ones with increasing indices) suffices. But there's no harm in summing over <u>every</u> I if you want (and it simplifies things later).

We now think about notation.

ω : a vector valued function

$$\vec{x} \longmapsto \underbrace{\sum_I f_I(\vec{x})\,dx_I}_{\text{in } \Lambda^k(\mathbb{R}^n)}$$

$\vec{x}$ in $\mathcal{U}$

(*)

$\omega(\vec{x})$: an element of $\Lambda^k(\mathbb{R}^n)$, an alternating, multilinear function from $(\mathbb{R}^n)^k \to \mathbb{R}$

$\omega(\vec{x})(\vec{v}_1, \ldots, \vec{v}_k)$: a real number.

(*) When $k=0$, $\omega(\vec{x})$ is an element of $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$.

Examples.

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then every $\omega \in A^1(\mathbb{R}^2)$ can be written in the form:

$$\omega = f_1 \, dx_1 + f_2 \, dx_2$$

So if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ we have

$$\begin{aligned} \omega(\vec{x})(\vec{v}) &= f_1(\vec{x}) \, dx_1(\vec{v}) \\ &\quad + f_2(\vec{x}) \, dx_2(\vec{v}) \\ &= f_1(\vec{x}) \cdot v_1 + f_2(\vec{x}) \cdot v_2 \\ &= \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R} \end{aligned}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then every $\omega \in A^2(\mathbb{R}^3)$ can be written in the form:

$$\omega = f_{12}\, dx_1 \wedge dx_2 + f_{13}\, dx_1 \wedge dx_3 + f_{23}\, dx_2 \wedge dx_3$$

So if we have vectors $\vec{v}_1, \vec{v}_2$ the value

$$\omega(\vec{x})(\vec{v}_1, \vec{v}_2) = f_{12}(\vec{x})\, dx_1 \wedge dx_2(\vec{v}_1, \vec{v}_2) + f_{13}(\vec{x})\, dx_1 \wedge dx_3(\vec{v}_1, \vec{v}_2) + f_{23}(\vec{x})\, dx_2 \wedge dx_3(\vec{v}_1, \vec{v}_2)$$

$$= f_{12}(\vec{x}) \cdot \det \begin{bmatrix} (\vec{v}_1)_1 & (\vec{v}_2)_1 \\ (\vec{v}_1)_2 & (\vec{v}_2)_2 \end{bmatrix}$$

$$+ f_{13}(\vec{x}) \cdot \det \begin{bmatrix} (\vec{v}_1)_1 & (\vec{v}_2)_1 \\ (\vec{v}_1)_3 & (\vec{v}_2)_3 \end{bmatrix}$$

$$+ f_{23}(\vec{x}) \cdot \det \begin{bmatrix} (\vec{v}_1)_2 & (\vec{v}_2)_2 \\ (\vec{v}_1)_3 & (\vec{v}_2)_3 \end{bmatrix}$$

which is starting to look a <u>lot</u> like expansion by minors in a determinant!

In fact, this <u>is</u> a determinant:

$$\omega(\vec{x})(\vec{v}_1, \vec{v}_2) =$$

$$\det \begin{bmatrix} f_{23}(\vec{x}) & \uparrow & \uparrow \\ -f_{13}(\vec{x}) & \vec{v}_1 & \vec{v}_2 \\ f_{12}(\vec{x}) & \downarrow & \downarrow \end{bmatrix}$$

Note that it's not traditional to write forms with a "vector" superscript, but we certainly <u>could</u>: $\omega(\vec{x})$ <u>is</u> a vector (in a vector space of functions) (so it's also a function).

We define the wedge product for differential forms by bilinearity of wedge, so

Definition. If $\omega \in A^k(\mathcal{U}), \eta \in A^\ell(\mathcal{U})$ for some open $\mathcal{U} \subset \mathbb{R}^n$, and

$$\omega = \sum_I f_I \, dx_I \quad \text{while}$$

$$\eta = \sum_J g_J \, dx_J, \quad \text{we define}$$

$$\omega \wedge \eta = \sum_{I,J} f_I g_J \, dx_I \wedge dx_J$$

$$= \sum_{I,J} f_I g_J \, dx_{IJ}$$

Proposition. If $U \subset \mathbb{R}^n$ is open, $\omega \in A^k(U)$, $\eta \in A^\ell(U)$, $\varphi \in A^m(U)$, then

1) If $k = \ell = m$, we have

$$\omega + \eta = \eta + \omega \in A^k(U)$$

$$(\omega + \eta) + \varphi = \omega + (\eta + \varphi) \in A^k(U)$$

2) $$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega \in A^{k+\ell}(U)$$

3) $$(\omega \wedge \eta) \wedge \varphi = \omega \wedge (\eta \wedge \varphi) \in A^{k+\ell+m}(U)$$

4) If $k = \ell$, then

$$(\omega + \eta) \wedge \varphi = \omega \wedge \varphi + \eta \wedge \varphi$$

All of this seems awfully simple and clean; so much that you might worry that it doesn't tell you much. But there are some nonobvious conclusions already.

Lemma. If $k>n$, then $A^k(\mathbb{R}^n)$ contains only one form $\omega=0$.

Proof. The vector space $\Lambda^k(\mathbb{R}^n)$ has a basis consisting of functions dx_I where $I=(i_1,\dots,i_k)$. Since $k>n$, an index must be repeated. But then $dx_I(\vec{v}_1, \ldots, \vec{v}_k) = 0$ for all inputs. Thus $\Lambda^k(\mathbb{R}^n) = \{0\}$.

Now ω is a smooth map $\omega : \mathcal{U} \to \{0\}$, so ω must have the constant value 0. This uniquely determines ω, so there is only a single element in $\mathcal{A}^k(\mathbb{R}^n)$. $\square$

Differentiating differential forms.

Consider the 0-form. By our definition, if $U \subset \mathbb{R}^n$ is open

$$A^0(U) = \text{smooth functions from } U \to \Lambda^0(\mathbb{R}^n) = \mathbb{R}$$

Now we know how to define the derivative: If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ then each $Df(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$ is a linear function from $\mathbb{R}^n \to \mathbb{R}$.

If we instead think of $Df(\vec{x})$ as an element of $\Lambda^1(\mathbb{R}^n)$, we are inspired to say

Definition. The <u>exterior</u> <u>derivative</u> of $f \in A^0(U)$ is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n \in A^1(U)$$

If $\omega = \sum_I f_I \, dx_I \in A^k(U)$, then

$$d\omega = \sum_I df_I \wedge dx_I \in A^{k+1}(U).$$

We note that

$$d\omega = \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Examples.

$$f : \mathbb{R} \to \mathbb{R} . \qquad df = f'(x)\, dx$$

$$\omega = y\, dx + x\, dy \in A^1(\mathbb{R}^2)$$

$$d\omega = \left(\overset{0}{\cancel{\frac{\partial}{\partial x} y}}\, dx + \overset{1}{\cancel{\frac{\partial}{\partial y} y}}\, dy \right) \wedge dx$$

$$+ \left(\overset{1}{\cancel{\frac{\partial}{\partial x} x}}\, dx + \overset{0}{\cancel{\frac{\partial}{\partial y} x}}\, dy \right) \wedge dy$$

$$= dy \wedge dx + dx \wedge dy = 0.$$

$$\omega = x_1\, dx_2 + x_3\, dx_4 + x_5\, dx_6 \in A^1(\mathbb{R}^6)$$

$$d\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 \in A^2(\mathbb{R}^6)$$

Proposition. Let $\omega \in A^k(U)$, $\eta \in A^\ell(U)$ and $f \in A^0(U)$ a smooth function.

1) When $k = \ell$,

$$d(\omega + \eta) = d\omega + d\eta$$

2) $d(f\omega) = df \wedge \omega + f\,d\omega$

3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

4) $d(d\omega) = 0$.

Proof. We start with 1. Suppose f and g are smooth functions $U \to \mathbb{R}$.

$$d(f+g) = \sum_i \frac{\partial}{\partial x_i}(f+g)\,dx_i$$

$$= \sum_i \frac{\partial}{\partial x_i} f\,dx_i + \frac{\partial}{\partial x_i} g\,dx_i$$

$$= df + dg$$

Now suppose $\omega = \sum_I f_I \, dx_I$

and $\eta = \sum_I g_I \, dx \in A^k(U)$. $\circledast$

Then

$$d(\omega + \eta) = d\left(\sum_I (f_I + g_I)\, dx_I\right)$$

(definition of d) $$= \sum_I d(f_I + g_I) \wedge dx_I$$

(we just proved it) $$= \sum_I (df_I + dg_I) \wedge dx_I$$

(linearity of $\wedge$) $$= \sum_I df_I \wedge dx_I + dg_I \wedge dx_I$$

(rearranging sums) $$= d\omega + d\eta.$$

$\circledast$ Since these are both k-forms, we are summing over the <u>same</u> set of increasing k-indices for $\mathbb{R}^n$, so it makes sense to write $\sum_I$ in each case.

Now we'll prove property 3.

We have proved that d is linear and know that $\wedge$ is bilinear.

So both sides of the <u>proposed</u> equation

$$d(\omega \wedge \eta) \overset{?}{=} d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

are linear in ω and in η.

Therefore, it suffices to show

$$d(f\,dx_I \wedge g\,dx_J)$$

$$= d(fg\,dx_I \wedge dx_J)$$

$$= d(fg) \wedge dx_I \wedge dx_J$$

$$= (g\,df + f\,dg) \wedge dx_I \wedge dx_J$$

$$= g\, df \wedge dx_I \wedge dx_J$$
$$+ f\, dg \wedge dx_I \wedge dx_J$$
$$= (df \wedge dx_I) \wedge (g\, dx_J)$$
$$+ (-1)^{k \cdot 1} f\, dx_I \wedge dg \wedge dx_J$$
$$= d(f\, dx_I) \wedge g\, dx_J$$
$$+ (-1)^k f\, dx_I \wedge d(g\, dx_J).$$

Note that property 2 follows by the same argument.

We now consider property 4:

$$d(d(\omega)) =$$

Since d is linear, $d \circ d$ is linear. So it suffices to show this in the case $\omega = f\, dx_I$.

$$d\omega = df \wedge dx_I$$
$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

so

$$d(d\omega) = \sum_{j=1}^{n} d\left(\frac{\partial f}{\partial x_j}\right) \wedge dx_j \wedge dx_I$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} dx_k \wedge dx_j \wedge dx_I$$

Now $dx_k \wedge dx_j = (-1)^{1 \cdot 1} dx_j \wedge dx_k$.

So we may regroup this sum as

$$= \sum_{j=1}^{n} \sum_{k=1}^{j-1} \underbrace{\left(\frac{\partial^2 f}{\partial x_j \partial x_k} - \frac{\partial^2 f}{\partial x_k \partial x_j} \right)}_{\substack{=0, \text{ since the functions are smooth } (C^\infty) \\ \text{they are } C^2 \text{ and mixed partials commute.}}} dx_j \wedge dx_k \wedge dx_I$$

This completes the proof.

Pullbacks.

Definition. Let $\mathcal{U} \subset \mathbb{R}^m$ be open and $\vec{g}: \mathcal{U} \to \mathbb{R}^n$ be smooth. If $\omega \in A^k(\mathbb{R}^n)$, we define the <u>pullback</u> $\vec{g}^* \omega \in A^k(\mathcal{U})$ as follows:

1) If ω is a 0-form, then $g^*\omega : \mathcal{U} \to \mathbb{R}$ is given by $\omega \circ \vec{g}$.

2) Suppose $u_1, \ldots, u_m$ are coordinates for $\mathbb{R}^m$. We define

$$g^* dx_i = dg_i = \sum_{j=1}^{m} \frac{\partial g_i}{\partial u_j} du_j$$

where g_i is the ith coordinate function for $\vec{g}: \mathcal{U} \subset \mathbb{R}^m \to \mathbb{R}^n$.

3) We define

$$g^*(\omega \wedge \eta) = g^*\omega \wedge g^*\eta$$

so

$$g^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = dg_{i_1} \wedge \cdots \wedge dg_{i_k}$$

For convenience, we use dg_I to denote $g^*(dx_I)$.

4) We define $g^*(\omega + \eta) = g^*\omega + g^*\eta$.

Putting these together, we see that if $\omega = \sum_I f_I \, dx_I \in A^k(\mathbb{R}^n)$ we have

$$g^*\omega = \sum_I f_I \circ g \, dg_I$$

$$= \sum_I (f_I \circ g) \, dg_{i_1} \wedge \cdots \wedge dg_{i_k}.$$

This is really technical and dry. But let's observe that we <u>can</u> use it to do calculations.

Examples.

If $g: U \subset \mathbb{R} \to \mathbb{R}$, and $f: \mathbb{R} \to \mathbb{R}$,

$$g^*(f\,dx) = (f \circ g)\,dg$$
$$= f \circ g \,\frac{\partial g}{\partial u}\,du$$

or

$$\left(g^*(f\,dx)\right)(u) = f(g(u)) \cdot g'(u)\,du$$

a 1-form on $U \subset \mathbb{R}$

Example.

Suppose $g : \mathbb{R} \to \mathbb{R}^2$ is given by

$$\vec{g}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

Now

$$Dg(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial t} \end{bmatrix}$$

so

$$g^* dx = -\sin t \, dt$$

$$g^* dy = \cos t \, dt$$

Now suppose $\omega \in A^1(\mathbb{R}^2)$ is

$$\omega = -y \, dx + x \, dy$$

Then

$$g^*\omega = \underbrace{(-\sin t)}_{f_1(g(t))}\overbrace{(-\sin t\, dt)}^{dg_1}$$

$$+ \underbrace{(\cos t)}_{f_2(g(t))}\overbrace{(\cos t\, dt)}^{dg_2}$$

$$= \sin^2 t\, dt + \cos^2 t\, dt$$

$$= dt$$

Example. Let $g: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\vec{g}\left(\begin{bmatrix} r \\ \theta \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

we compute

$$D\vec{g}\left(\begin{bmatrix} r \\ \theta \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & -r\cos\theta \end{bmatrix}$$

So we see

$$g^* dx = \cos\theta \, dr - r\sin\theta \, d\theta$$

$$g^* dy = \sin\theta \, dr + r\cos\theta \, d\theta$$

and

$$g^*(dx \wedge dy) = g^*dx \wedge g^*dy$$

$$\begin{aligned} &= \cos\theta \sin\theta \, \cancel{dr \wedge dr}^{\,0} \\ &\quad + r\cos^2\theta \, dr \wedge d\theta \\ &\quad - r\sin^2\theta \, d\theta \wedge dr \\ &\quad - r^2 \sin\theta \cos\theta \, \cancel{d\theta \wedge d\theta}^{\,0} \end{aligned}$$

$$= \left(r\cos^2\theta + r\sin^2\theta \right) dr \wedge d\theta$$

$$= r \, dr \wedge d\theta$$

Now suppose $\omega = x\,dx + y\,dy \in A^1(\mathbb{R}^2)$.

Then we have

$$g^*\omega = g^*(x\,dx) + g^*(y\,dy)$$

$$= r\cos\theta \; g^*dx + r\sin y \; g^*dy$$

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x$ composed with g $\qquad$ $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = y$ composed with g

$$= r\cos\theta\,(\cos\theta\,dr - r\sin\theta\,d\theta)$$
$$+ r\sin\theta\,(\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$= (r\cos^2\theta + r\sin^2\theta)\,dr$$
$$\left(\cancelto{0}{-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}\right)d\theta$$

$$= r\,dr$$

As Shifrin points out, this process of simplifying wedge products when evaluating pullbacks naturally leads to determinants of submatrices of the derivative matrix.

Theorem. If $I=(i_1,\dots,i_k)$ is a k index for $\mathbb{R}^n$ and $g: U \subset \mathbb{R}^m \to \mathbb{R}^n$ then

$$g^* dx_I = \sum_{\substack{J \text{ in} \\ \text{increasing} \\ k\text{-indices} \\ \text{for } \mathbb{R}^m}} \det(D\vec{g})_{I,J}\, du_J$$

where $A_{I,J}$ denotes the submatrix obtained by choosing rows from I and columns from J.

Proof. By definition

$$g^*(dx_I) = dg_{i_1} \wedge \cdots \wedge dg_{i_k}$$

where $dg_{i_p} = \sum_{q=1}^{m} \frac{\partial g_{i_p}}{\partial u_q} du_q$. Now the wedge product is multilinear, so we can expand to get

$$dg_{i_1} \wedge \cdots \wedge dg_{i_k} =$$

$$= \sum_{\substack{J \text{ a } k\text{-index} \\ \text{for } \mathbb{R}^m}} \frac{\partial g_{i_1}}{\partial u_{j_1}} \cdots \frac{\partial g_{i_k}}{\partial u_{j_k}} du_{j_1} \wedge \cdots \wedge du_{j_k}$$

$$= \sum_{\substack{J \text{ a } k\text{-index} \\ \text{for } \mathbb{R}^m}} \left(\frac{\partial g_{i_1}}{\partial u_{j_1}} \cdots \frac{\partial g_{i_k}}{\partial u_{j_k}} \right) du_J$$

Now this is a sum over <u>all</u> k-indices J. However, if there's a repeated number in J, $du_J = 0$. So we may as well sum over k-indices with distinct entries $j_1, \dots, j_k$. Any such index is a permutation σ of an increasing index J^+, and $du_J = (\text{sign } \sigma)\, du_{J^+}$. Collecting terms,

$$= \sum_{\substack{J^+ \text{ an} \\ \text{increasing} \\ k\text{-index}}} \underbrace{\left(\sum_{\substack{\text{permutations} \\ \sigma \text{ of } \{1,\dots,k\}}} \text{sign}(\sigma) \frac{\partial g_{i_1}}{\partial u_{\sigma(j_1^+)}} \cdots \frac{\partial g_{i_k}}{\partial u_{\sigma(j_k^+)}} \right)}_{\det (Dg)_{I, J^+}} du_{J^+}$$

This algebraic miracle completes proof. $\square$

Proposition. Let $U \subset \mathbb{R}^m$ be an open set and $\vec{g}: U \to \mathbb{R}^n$ be a smooth function. If $\omega \in \Lambda^k(U)$,

$$\vec{g}^*(d\omega) = d(\vec{g}^*(\omega)).$$

Proof. Suppose $k=0$. Then $\omega = f$ where $f: \mathbb{R}^n \to \mathbb{R}$ and we compute

$$d(\vec{g}^* f) = d(f \circ \vec{g})$$
$$= \sum_{j=1}^{m} \frac{\partial (f \circ \vec{g})}{\partial u_j} du_j$$

Now the function $f \circ g: U \to \mathbb{R}$. The partial derivative $\frac{\partial (f \circ \vec{g})}{\partial u_j}$ is the jth component of the $1 \times m$ matrix $D(f \circ \vec{g})$, which (by chain rule) is

$$\underbrace{D(f \circ \vec{g})(\vec{u})}_{1 \times m} = \underbrace{Df(\vec{g}(\vec{u}))}_{1 \times n} \cdot \underbrace{D\vec{g}(\vec{u})}_{n \times m}$$

Now the ij-entry

$$\left(Df(\vec{g}(\vec{u}))D\vec{g}(\vec{u})\right)_{ij} = \sum_{k=1}^{n} Df(\vec{g}(\vec{u}))_{ik}\, D\vec{g}(\vec{u})_{kj}$$

So the $1j$ entry $\frac{\partial(f\circ g)}{\partial u_j}$ is given by

$$\frac{\partial(f\circ g)}{\partial u_j} = \sum_{k=1}^{n} Df(\vec{g}(\vec{u}))_{1k}\, D\vec{g}(\vec{u})_{kj}$$

$$= \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\vec{g}(\vec{u})) \cdot \frac{\partial g_k}{\partial u_j}(\vec{u})$$

Thus

$$\sum_{j=1}^{m} \frac{\partial(f\circ \vec{g})}{\partial u_j}\, du_j = \sum_{j=1}^{m}\left(\sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\vec{g}(\vec{u}))\cdot\frac{\partial g_k}{\partial u_j}\right) du_j$$

reversed order of sums

$$= \sum_{k=1}^{n}\left(\sum_{j=1}^{m} \frac{\partial f}{\partial x_k}(\vec{g}(\vec{u}))\cdot\frac{\partial g_k}{\partial u_j}\, du_j\right)$$

pulled $\frac{\partial f}{\partial x_k}$ out of j sum

$$= \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}(\vec{g}(\vec{u})) \sum_{j=1}^{m} \frac{\partial g_k}{\partial u_j} du_j \right)$$

$$= \sum_{k=1}^{n} \vec{g}^{*}\left(\frac{\partial f}{\partial x_k}\right) \vec{g}^{*} dx_k$$

pullback is composition for functions $\because$ $\frac{\partial f}{\partial x_k}$ is a scalar function

definition of pullback for basis 1-form dx_k

$$= \sum_{k=1}^{n} \vec{g}^{*}\left(\frac{\partial f}{\partial x_k} dx_k\right)$$

$\frac{\partial f}{\partial x_k} dx_k = \frac{\partial f}{\partial x_k} \wedge dx_k$

b/c $\frac{\partial f}{\partial x_k}$ is a 0-form, and

$\vec{g}^{*}(\omega \wedge \eta) = \vec{g}^{*}(\omega) \wedge \vec{g}^{*}(\eta)$

$$= \vec{g}^*\left(\sum_{k=1}^{n} \frac{\partial f}{\partial x_k} dx_k\right)$$

pollback is linear

$$= \vec{g}^*(df)$$

definition of df

Now we get to integration.

Definition. The vector space $\Lambda^n(\mathbb{R}^n)$ is 1-dimensional, and we may take

$$dx_{1\cdots n} = dx_1 \wedge \cdots \wedge dx_n = \det$$

as the basis element. We call the corresponding constant form "the volume form" or "the standard n-form".

Definition. If $\omega \in \Lambda^n(\mathcal{U})$, $\mathcal{U} \subset \mathbb{R}^n$ is an open set, and $\Omega \subset \mathcal{U}$ is a region, and $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$, we <u>define</u>

$$\int_\Omega \omega := \int_\Omega f \, dArea$$

Note: we are only dealing with <u>smooth</u> forms, so f is a smooth function. Thus f is continuous and therefore integrable.

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a region and $\vec{g}: \Omega \to \mathbb{R}^n$ be smooth and 1-1 with $\det D\vec{g}(\vec{x}) > 0$. Then for any n-form $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ on $S = \vec{g}(\Omega)$

$$\int_S \omega = \int_\Omega \vec{g}^{\,*}\omega.$$

Proof. We have already shown

Theorem. If $I=(i_1,\dots,i_k)$ is a k index for $\mathbb{R}^n$ and $g: U \subset \mathbb{R}^m \to \mathbb{R}^n$ then

$$\vec{g}^* dx_I = \sum_{\substack{J \text{ in} \\ \text{increasing} \\ k\text{-indices} \\ \text{for } \mathbb{R}^m}} \det (D\vec{g})_{I,J} \, du_J$$

If $k=n$, there's only one increasing k-index: $1,\dots,n$, and so

$$\vec{g}^* dx_{1\cdots n} = \det D\vec{g}_{1\dots n, 1 \dots n} \, du_{1\cdots n}$$

$$= \det D\vec{g} \, du_{1\cdots n}.$$

Now

$$\vec{g}^* f\omega = \vec{g}^* f \, \vec{g}^* dx_{1\cdots n}$$

$$= (f\circ\vec{g}) \det D\vec{g} \, du_{1\cdots n}$$

But then

$$\int_{\Omega} \vec{g}^{*}\omega = \int_{\Omega} f\circ\vec{g}\ \det D\vec{g}\ du_{1\ldots n}$$

det $D\vec{g} > 0$ by hypothesis

$$= \int_{\Omega} f\circ\vec{g}\ |\det D\vec{g}|\ d\text{Area}$$

definition of integration of n-forms

change of variables thm

$$= \int_{\vec{g}(\Omega)} f\ d\text{Area}$$

definition of integration of n-forms

$$= \int_{S(\Omega)} f\ dx_1\wedge\cdots\wedge dx_n$$

definition of ω

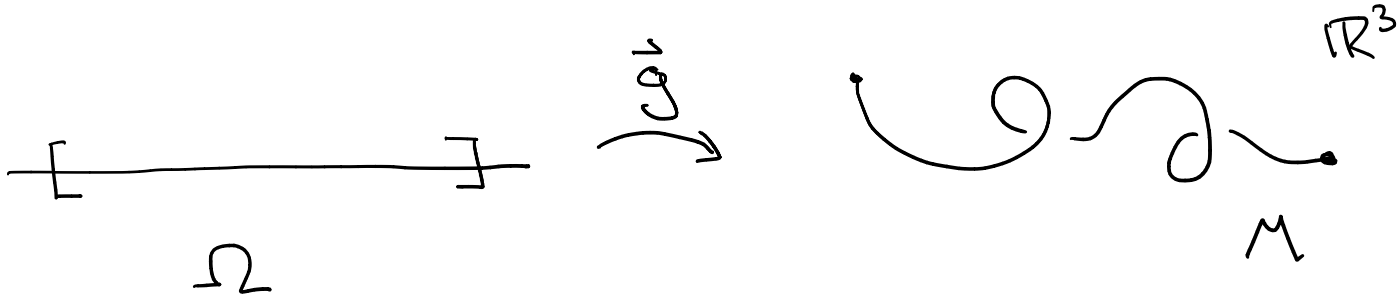
$$= \int_{S(\Omega)} \omega$$

This was kind of the point of forms!

It allows to engage with this question: what does it mean to integrate over a (smooth) submanifold of $\mathbb{R}^n$?

M is not a region (and anyway has volume zero), so $\int_M f \, dArea_{\mathbb{R}^3}$ is not defined.

For now, we define a <u>parametrized K-dimensional submanifold of $\mathbb{R}^n$</u> to be a subset $M \subset \mathbb{R}^n$ so that $\exists$ a region $\Omega \subset \mathbb{R}^K$ and a smooth 1-1 map $\vec{g}: \Omega \to \mathbb{R}^n$ with rank $Dg = K$ on Ω so that $M = g(\Omega)$.

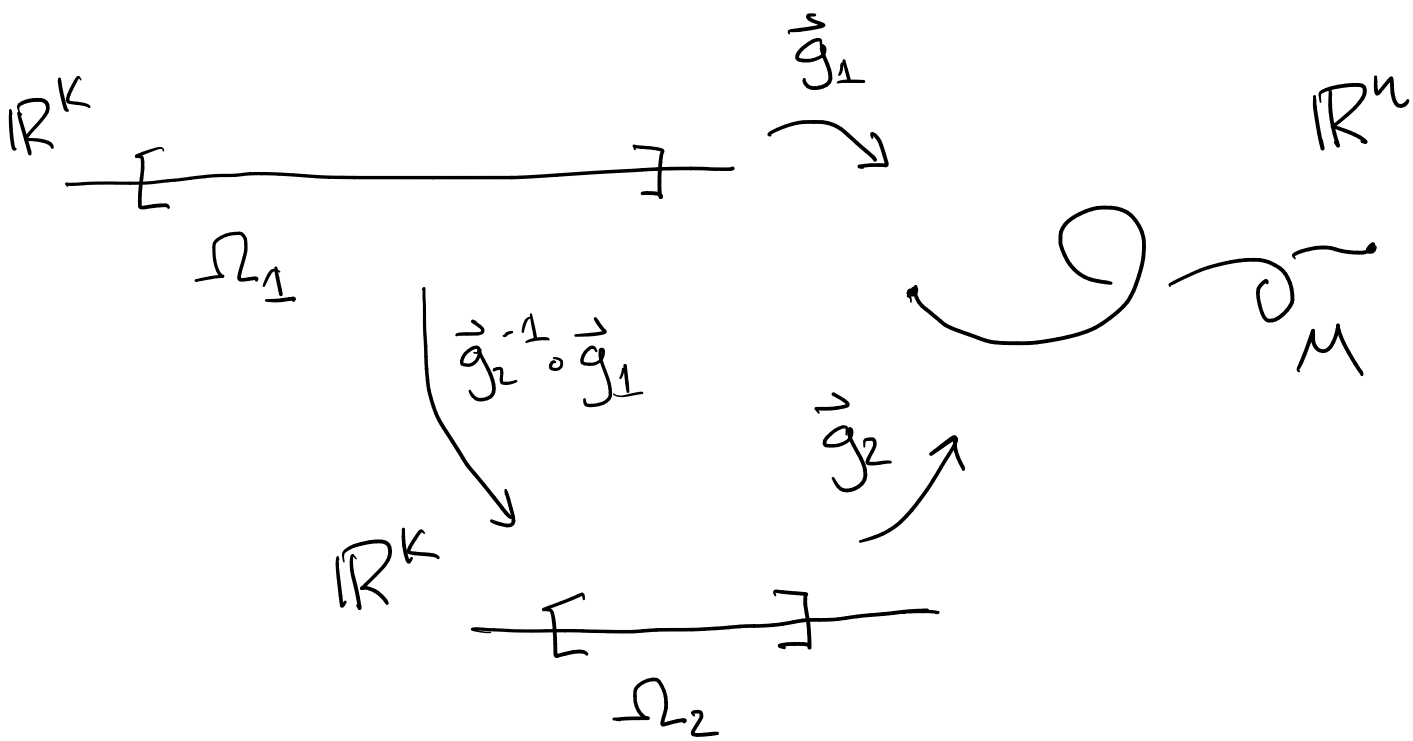


Now suppose $\omega \in A^k(\mathbb{R}^n)$. Then we <u>define</u>

$$\int_M \omega = \int_\Omega \vec{g}^* \omega$$

an integral of a k-form over a region in $\mathbb{R}^k$ defined previously!

The parametrization $\vec{g}$ is certainly not unique. So how does the choice of $\vec{g}$ affect the integral?



Now we know $\vec{g}_2^{-1} \circ \vec{g}_1 : \Omega_1 \to \Omega_2$.

So we may write

pullback integration theorem!

$$\int_{\Omega_2} \vec{g}_2^{\,*} \omega = \pm \int_{\Omega_1} (\vec{g}_2^{-1} \circ \vec{g}_1)^* \, \vec{g}_2^{\,*} \omega$$

the sign depends on whether $\det D(\vec{g}_2^{-1} \circ \vec{g}_1) > 0$ or not

we'll prove $(f \circ g)^* = g^* f^*$ in homework

$$= \pm \int_{\Omega_1} (\vec{g}_2 \circ \vec{g}_2^{-1} \circ \vec{g}_1)^* \omega$$

$$= \pm \int_{\Omega_1} \vec{g}_1^{\,*} \omega$$

and we see that the integral is the same for all parametrizations with $\det(D(\vec{g}_2^{-1} \circ g_1)) > 0$.