8.3 Line Integrals and Green's Theorem
Definition. A vector field \( \vec{F} \) on an open set \( U \subset \mathbb{R}^n \) is a function \( \vec{F} : U \to \mathbb{R}^n \) which associates a vector to each point in \( U \).

Now we previously (curves, 3.5) defined a parametrized curve to be a map \( \vec{g} : [a, b] \to \mathbb{R}^n \). Recall that

\[
T(t) = \frac{\vec{g}'(t)}{||\vec{g}'(t)||}
\]

is called the unit tangent vector to \( \vec{g} \) at \( \vec{g}(t) \).
and that we called a parametrization regular when $\|\hat{g}'(t)\| = 0$ (so $T$ is well-defined). Looking back on this, we now say "a parametrization is regular when rank $Dg = 1$.

Construction. Every 1-form on $U \subset \mathbb{R}^n$ $\omega$ has a corresponding vector field $\vec{F}$ so that

$$\omega = \sum F_i \, dx_i \iff \vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$$

Then if $C \subset \mathbb{R}^n = \hat{g}([a, b])$,

$$\int_C \omega = \int_{\hat{g}} \omega = \int_a^b \left( \sum_{i=1}^n F_i(\hat{g}(t)) \hat{g}_i(t) \right) dt$$

pullback of $\vec{F}_i$

pullback of $dx_i$
\[
= \int_{[a,b]} \hat{F}(\hat{g}(t)) \cdot \hat{g}'(t) \, dt
\]

This is called a "line integral" or "path integral."

If \( \|\hat{g}'(t)\| = 1 \) for all \( t \in [a, b] \), we say "\( g \) is parametrized by arclength" and use \( s \) for the parameter. This motivates us to write our path integral as

\[
= \int_{[a,b]} \hat{F}(\hat{g}(t)) \cdot \frac{\hat{g}'(t)}{\|\hat{g}'(t)\|} \|\hat{g}'(t)\| \, dt
\]

\[
= \int_{C} \hat{F} \cdot T \, ds
\]

\text{definitions}
We've been a bit informal about the smoothness of \( \hat{g} \). Everything makes sense as long as \( C \) is parametrized by a finite collection of \( C^1 \) maps

\[
\hat{g}_1 : [a_1, b_1] \rightarrow \mathbb{R}^n \\
\hat{g}_2 : [a_3, b_3] \rightarrow \mathbb{R}^n
\]

where \( \hat{g}_j(b_j) = \hat{g}_{j+4}(a_{j+1}) \). We call these curves "piecewise \( C^1 \)."
Proposition. If $C$ is a curve in $\mathbb{R}^n$ parametrized by $\tilde{g}: [a, b] \to \mathbb{R}^n$, let $C^-$ be the curve parametrized by $\tilde{h}: [\tilde{a}, \tilde{b}] \to \mathbb{R}^n$, $\tilde{h}(u) = \tilde{g}(a+b-u)$. Then for every $w \in A^1(\mathbb{R}^n)$, we have

$$\int w = -\int_{C^-} w$$

Proof. We write

$$\int_{C^-} w = \int_{a}^{b} \tilde{h}^* w = \int_{a}^{b} F(\tilde{h}(u)) \cdot \tilde{h}'(u) \, du$$

$$= \int_{a}^{b} F(\tilde{g}(a+b-u)) \cdot (-\tilde{g}'(a+b-u)) \, du$$
\[- \int_{a}^{b} F(\mathbf{g}(t)) \cdot \mathbf{g}'(t) \, dt \quad \text{where } t = a + b - u\]

\[= - \int_{a,b} \mathbf{g}^* \omega.\]

\[= - \int_{c} \omega.\]

Example. Let $C$ be the line segment $[\frac{1}{2}, \frac{1}{2}]$ to $[\frac{2}{2}, \frac{2}{2}]$ and let $\omega = xy \, dz$. To compute $\int_{C} \omega$, we parametrize $C$ by

\[\mathbf{g}(t) = (1-t)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\]

where $t \in [0,1]$. 
We then have
\[ \hat{g}'(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \]

and can compute
\[
\int_c \omega = \int_c \hat{g}^* \omega = \int_{[0,1]} \\
= \int_0^1 \left( (1-t)1 + t \cdot 2 \right) \left( (1-t)(-1) + t \cdot 2 \right) \, 2 \, dt \\
= \int_0^1 \left( 1 + t \right) \left( -1 + 3t \right) \, 2 \, dt \\
= \int_0^1 \left( 6t^2 + 4t - 2 \right) \, dt = 2.
\]
Definition. If \( \hat{F}(x): \mathbb{R}^n \to \mathbb{R}^n \) is a vector field whose value is a force at each point in space, and \( \omega \) is the corresponding 1-form, the work done by the force field by a particle moving along a path \( C \) is

\[
\text{work} = \int_C \omega
\]

Definition. If a particle of mass \( m \) has velocity vector \( \hat{v} \), the kinetic energy \( KE = \frac{1}{2}m\|\hat{v}\|^2 \).
We can now prove the Work-Energy Theorem. If the only force acting on a particle of mass \( m \) causes the particle to move along a path \( C \), then

\[
\text{work} = \text{change in kinetic energy}.
\]

**Proof.** Suppose the force is given by \( F(x) \) and the path by \( g(t) \).

\[
\text{work} = \int_C \omega = \int_a^b F(g(t)) \cdot \dot{g}'(t) \, dt
\]

\[
= \int_a^b m \dddot{g}(t) \cdot \dot{g}'(t) \, dt
\]

\[
= \int_a^b \frac{d}{dt} \left( m \frac{d}{dt} (\| \dot{g}'(t) \|^2) \right) \, dt
\]
\[ \frac{1}{2} m \left( \| \dot{\vec{g}}'(b) \|^2 - \| \dot{\vec{g}}'(a) \|^2 \right) \]

= change in kinetic energy. \( \square \)

Now we can prove the fundamental theorem of calculus for line integrals.

Proposition. If \( \omega = df \in \mathcal{A}^1(\mathbb{R}^n) \) and \( C \) is a path from \( \vec{a} \) to \( \vec{b} \) in \( \mathbb{R}^n \),

\[ \int_C \omega = f(\vec{b}) - f(\vec{a}). \]

Proof. Suppose \( C \) is parametrized by \( \dot{\vec{g}} \).

\[ \int_C \omega = \int_a^b \dot{\vec{g}}^* \omega = \int_a^b \dot{\vec{g}}^* (df) \]

\[ = \int_a^b d(\dot{\vec{g}}^* f) = \int_a^b d(f \circ \dot{\vec{g}}) \]
\[ = \int_{a}^{b} (f \circ \dot{g})'(t) \, dt \]
\[ = (f \circ \dot{g})(b) - (f \circ \dot{g})(a) \]
\[ = f(\dot{g}(b)) - f(\dot{g}(a)) \]
\[ = f(b) - f(a). \]

**Corollary.** If \( \tilde{F}(\dot{x}) = \nabla f(\dot{x}) \), then
\[ \int_{\gamma} \tilde{F} \cdot \dot{F} \, ds = \tilde{F}(b) - \tilde{F}(a). \]

Notice that the value of the integral doesn’t depend on the path!

**Definition.** If \( \omega = d\eta \), we say that \( \eta \) is a **potential form** for \( \omega \).

(Or a potential function if \( \eta \in \mathcal{A}(U) \).)
Theorem. Let \( \omega = \sum F_i \, dx_i \in A^1(U) \) with \( U \subset \mathbb{R}^n \). The following are equivalent:

1) For every closed path \( C \subset U \),
\[
\int_C \omega = 0.
\]

2) If \( \vec{a} \) and \( \vec{b} \) are joined by paths \( C \subset U \) and \( C' \subset U \),
\[
\int_C \omega = \int_{C'} \omega
\]
(In this case, we say the integral is path independent and write \( \int_{\vec{a}}^{\vec{b}} \omega = \int_C \omega \) for any \( C \) which starts at \( \vec{a} \) and ends at \( \vec{b} \).)

3) \( \omega = df \) for some potential function \( f : U \to \mathbb{R} \).
Note. If a force field $\mathbf{F} = \nabla f$, we say $\mathbf{F}$ is conservative. If so, and $\mathbf{C}$ is a path parametrized by $\mathbf{\hat{g}}$, 

$$\int_{\mathbf{C}} \mathbf{w} = \int_{a}^{b} \mathbf{\hat{F}}(\mathbf{\hat{g}}(t)) \cdot \mathbf{\hat{g}}'(t) \, dt$$

= work

$$= \frac{1}{2} m \| \mathbf{\hat{g}}'(b) \|^2 - \frac{1}{2} m \| \mathbf{\hat{g}}'(a) \|^2$$

(by work-energy theorem) but also

$$\int_{\mathbf{C}} \mathbf{w} = f(\mathbf{\hat{g}}(b)) - f(\mathbf{\hat{g}}(a))$$

(by fundamental theorem of calculus)

This leads physicists to call $-f(\mathbf{\hat{g}}(x))$ a potential energy for $\mathbf{\hat{F}}(x)$
So that they can write above as
\[ \Delta \text{K.E.} = - \Delta \text{P.E.} \Rightarrow \Delta (\text{K.E.} + \text{P.E.}) = 0 \]
and say "the sum of kinetic and potential energy is conserved."

Proof. (1 \Rightarrow 2)

If \(C_1, C_2\) are paths from \(\hat{a}\) to \(\hat{b}\), \(C_1 \cup C_2^-\) is a closed path, so

\[ 0 = \int \omega = \int \omega + \int \omega = \int \omega - \int \omega \]
\[ C_1 \cup C_2^- \quad C_1 \quad C_2^- \quad C_1 \quad C_2 \]
so \(\int_{C_1} \omega = \int_{C_2} \omega\).

(2 \Rightarrow 3) Fix any \(\tilde{a} \in \mathcal{U}\) and define
\[ f(\tilde{x}) = \int_{\tilde{a}}^{\tilde{x}} \omega \]
(by hypothesis, the path doesn't matter).

To prove $df = \omega$, we must show

$$F_i(x) = \frac{\partial f}{\partial x_i}(\hat{x})$$

$$= \lim_{h \to 0} \frac{f(\hat{x} + h\hat{e}_i) - f(\hat{x})}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{\hat{x}}^{\hat{x} + h\hat{e}_i} \omega$$

Now we can join $\hat{x}$ to $\hat{x} + h\hat{e}_i$ by $\hat{g} : [0, h] \to \mathbb{R}^n$, $\hat{g}(t) = \hat{x} + t\hat{e}_i$.

$$= \lim_{h \to 0} \frac{1}{h} \int_0^h \hat{g}^* \omega$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^h \sum_{j=1}^n F_j(\hat{g}(t)) \hat{g}^* dx_j$$

but

$$\hat{g}^* dx_j = g_j'(t) dt$$

where $g_j$ is the coordinate function.
But \( \tilde{g}(t) = \tilde{x} + t\tilde{e}_i \), so
\[
\tilde{g}'(t) = \delta_{ij}
\]
So we have
\[
= \lim_{h \to 0} \frac{1}{h} \int_0^h F_i(\tilde{g}(t)) \, dt
\]
\[
= \lim_{h \to 0} F_i(\tilde{g}(t_0)) \text{ for some } t_0 \in [0, h]
\]
by MVT for integrals
\[
= F_i(\tilde{g}(0)) \quad \text{continuity of } F_i
\]
\[
= F_i(\tilde{x}).
\]
which proves \( df = \omega \), as desired.
(3 \( \Rightarrow \) 4) Since \( \omega = df \), if \( C \) is closed
\[
\int_C \omega = \int_C df = f(\tilde{a}) - f(\tilde{b}) = 0. \quad \square
\]
Definition. If $\omega \in A^k(U)$ and $d\omega = 0$, we say $\omega$ is closed.

If $\omega = d\eta$ for some $\eta \in A^{k-1}(U)$ we say $\omega$ is exact.

Now if $\omega = df$, then $d\omega = d(df) = 0$.

So every exact form is closed. Is every closed form exact? The answer will be interesting.

Example. Suppose

$$\omega = (e^x + 2xy)\,dx + (x^2 + \cos y)\,dy$$

We would like to find a potential function $f$ so that $df = \omega$. 
Such a function (if it exists) has
\[ \frac{\partial f}{\partial x} = e^x + 2xy, \quad \frac{\partial f}{\partial y} = x^2 + \cos y. \]
We can find it by "partial integration"
\[ \int e^x + 2xy \, dx = e^x + x^2y + C(y) \]

Partial differentiation

\[ \frac{\partial}{\partial y} e^x + x^2y + C(y) = x^2 + C'(y) \]

Solve for \( C'(y) \)
\[ x^2 + \cos y = x^2 + C'(y) \]
\[ C'(y) = \cos y \]

Integrate again
\[ C(y) = \int \cos y \, dy = -\sin y + D \]
Assemble results:

\[ f(x, y) = e^x + x^2y - \sin y + D. \]

We want to prove a theorem about when this works, but need a tool. Suppose \( f : [a, b] \times [c, d] \to \mathbb{R} \) is \( C^1 \) and consider

\[
F(x) = \int_c^d f([x, y]) \, dy.
\]

You proved in homework that

\[
F'(x) = \frac{\partial}{\partial x} \int_c^d f([x, y]) \, dy
= \int_c^d \frac{\partial}{\partial x} f([x, y]) \, dy.
\]

This is called “differentiating under the integral sign.”
Definition. We say a region $\Omega \subset \mathbb{R}^n$ is starlike if there is some $\hat{a} \in \Omega$ so that for every $\hat{x} \in \Omega$, the line segment $\hat{a}\hat{x} \subset \Omega$.

\begin{center}
\begin{tabular}{cc}
\text{starlike} & \text{not starlike} \\
\end{tabular}
\end{center}

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a starlike region and $\omega \in A_0^1(\Omega)$. If $\omega$ is closed, then $\omega$ is exact.

Proof. Suppose $\omega = \sum F_i \, dx_i$. For any $\hat{x} \in \Omega$, we can parametrize the
line C from $\hat{a}$ to $\hat{x}$ by 
\[ \hat{g}(t) = \hat{a} + t(\hat{x} - \hat{a}), \quad t \in [0,1]. \]

We define 
\[ f(\hat{x}) = \int_{\hat{c}} w = \int_{[0,1]} \hat{g}^* w \]
\[ = \int_0^1 \sum_{j=1}^n F_j(\hat{g}(t)) g^j(t) \, dt \]

Now \[ \hat{g}'(t) = \hat{x} - \hat{a}, \]
so \[ g^j(t) = x_j - a_j. \]
\[ = \sum_{j=1}^n (x_j - a_j) \int_0^1 F_j(\hat{g}(t)) \, dt \]

We claim that $df = w$. So we have to compute
\[ \frac{\partial f}{\partial x_i} = \int_0^1 F_i (\dot{x}(t)) \, dt + \sum_{j=1}^n (x_j - a_j) \frac{\partial}{\partial x_i} \int_0^1 F_j (\dot{x}(t)) \, dt \]

\[ = \int_0^1 F_i (\dot{x}(t)) \, dt + \sum_{j=1}^n (x_j - a_j) \int_0^1 \frac{\partial}{\partial x_i} F_j (\dot{x}(t)) \, dt. \]

Now

\[ \frac{\partial}{\partial x_i} F_j (\dot{x}(t)) = \frac{\partial}{\partial x_i} F_j (\dot{x} + t(\ddot{x} - \dot{x})) \]

\[ = \frac{\partial F_j}{\partial x_i} (\dot{x} + t(\ddot{x} - \dot{x})). \frac{\partial}{\partial x_i} (\dot{x} + t(\ddot{x} - \dot{x})) \]

\[ = \frac{\partial F_j}{\partial x_i} (\dot{x}(t)) \cdot t \]
Now $w$ is closed, so $d\omega = 0$. But $d\omega = \sum_{1 \leq i,j \leq n} (\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}) dx_i \wedge dx_j$, so this means that $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$, and we can write

$$\int_0^1 t \frac{\partial F_j}{\partial x_i} (g(t)) dt = \int_0^1 t \frac{\partial F_i}{\partial x_j} (g(t)) dt$$

and we have

$$\sum_{j=1}^n (x_j - a_j) \int_0^1 \frac{\partial}{\partial x_i} F_j (g(t)) dt =$$

$$= \int_0^1 t \sum_{j=1}^n (x_j - a_j) \frac{\partial F_i}{\partial x_j} (g(t)) dt$$

the derivative

$$\frac{d}{dt} F_i (g(t))$$

by chain rule!
\[ \int_0^1 t (F_i \circ \hat{g})'(t) \, dt \]

\[ = \left( F_i \circ \hat{g} \right)(t) \bigg|_0^1 - \int_0^1 F_i(\hat{g}(t)) \, dt \]

\[ \text{integration by parts!} \]

\[ = F_i(\hat{g}(1)) - \int_0^1 F_i(\hat{g}(t)) \, dt \]

\[ = F_i(\mathbf{x}) - \int_0^1 F_i(\hat{g}(t)) \, dt \]

Thus

\[ \frac{\partial F}{\partial x_i} = \int_0^1 F_i(\hat{g}(t)) \, dt + \]

\[ + \sum_{j=1}^n (x_j - a_j) \int_0^1 \frac{\partial}{\partial x_i} F_j(\hat{g}(t)) \, dt \]

\[ = F_i(\mathbf{x}), \text{ as required. } \]
This theorem is usually enough: given a 1-form, check if it's closed, then try to construct a potential.

Example. Newton's law of gravitational says that the force of gravity exerted by a point mass $M$ at $\mathbf{r}$ is given by

$$\mathbf{F} = -GM \frac{\mathbf{r}}{||\mathbf{r}||^3}$$

The corresponding 1-form is

$$\omega = -GM \frac{x}{(x^2+y^2+z^2)^{3/2}} (xdx+ydy+zdz)$$

To find a potential, let's try

$$\int \frac{-x}{(x^2+y^2+z^2)^{3/2}} \, dx = \frac{1}{(x^2+y^2+z^2)^{1/2}} + C$$
and observe that this works, so

\[ f(\hat{x}) = \frac{GM}{\|\hat{x}\|} \]

is a potential function.

This means that

\[
\begin{align*}
\text{work} &= \int_{0}^{4} \hat{g} \cdot \omega \\
&= f(\hat{g}(4)) - f(\hat{g}(0)) \\
&= GM \left( \frac{1}{\|\hat{g}(4)\|} - \frac{1}{\|\hat{g}(0)\|} \right)
\end{align*}
\]
= change in kinetic energy

\[ = \frac{1}{2} \| \ddot{\mathbf{r}}(1) \|^2 - \frac{1}{2} \| \ddot{\mathbf{r}}(0) \|^2. \]

and we can see that an object in orbit is moving fastest when closest to the origin.

We can also see that \( \| \dot{\mathbf{r}}(t) \| \) is periodic - over a complete orbit no work is done, so the starting and ending kinetic energy are the same.
Green's Theorem on a Rectangle.
We have proved that if $\omega = df$, and $C$ is a curve from $a$ to $b$, we have $\int_C \omega = f(b) - f(a)$. This is a 1-d generalization of the fundamental theorem of calculus.
Let's try for 2d!

Theorem. (Green's theorem) Let $R \subseteq \mathbb{R}^2$ be a rectangle and let $\omega$ be a 1-form on $R$. Then if $\partial R$ is the closed curve given by following the boundary ccw,

$$\int_{\partial R} \omega = \int_R d\omega.$$
Proof. Suppose $R = [a,b] \times [c,d]$ and

$$\omega = P \, dx + Q \, dy$$

Now $d\omega = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \wedge dy$, and

$$\int_R d\omega = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, d\text{Area}$$

$$= \int_a^b \int_c^d \left( \int_a^b \frac{\partial Q}{\partial x} \, dx \right) \, dy - \int_a^b \int_c^d \left( \int_c^d \frac{\partial P}{\partial y} \, dy \right) \, dx$$

$$= \int_c^d \left( \int_a^b Q([x]) - Q([y]) \, dy \right) \, dx$$

$$- \int_a^b \left( \int_c^d P([x]) - P([y]) \, dx \right) \, dy$$

$$= \int_a^b P([x]) \, dx + \int_c^d Q([y]) \, dy$$

$$+ \int_a^c P([x]) \, dx + \int_d^b Q([y]) \, dy$$
Corollary. If $S \subset \mathbb{R}^2$ is parametrized by a rectangle and $\omega$ is a 1-form on $S$, then $\int_S \omega = \int_S d\omega$.

Proof.

\[ \int_S \omega = \int_R \hat{g}^* \omega = \int_R d(\hat{g}^* \omega) = \int_R \hat{g}^* (d\omega) = \int_S d\omega. \]

We observe that if we can "tile" a region by subsets parametrized by rectangles, the theorem works.
We note that sections of the boundary of the rectangles which map to the interior of $S$ cancel each other out, leaving only $\partial S$.

Example. Suppose that $\omega = -\frac{y}{2} \, dx + \frac{x}{2} \, dy$. Then $d\omega = -\frac{1}{2} \, dy \wedge dx + \frac{1}{2} \, dx \wedge dy = dx \wedge dy$. So

$$\text{area}(S) = \int_S dx \wedge dy = \int_S d\omega = \int_S \omega.$$
and so we can compute the area enclosed by any curve \( C \subset \mathbb{R}^2 \) by integrating

\[
\frac{1}{2} \int_C -y\,dx + x\,dy = \text{area enclosed!}
\]

Now we can say a bit more about closed and exact forms.

**Definition.** A subset \( X \subset \mathbb{R}^n \) is called **path-connected** if every \( a, b \in X \) are the endpoints of a curve \( C \subset X \).

**Definition.** A subset \( X \subset \mathbb{R}^n \) is called **simply connected** if it is path-connected and every closed curve \( C \subset X \) may be parametrized by \( \hat{g} : \mathbb{R} \to X \) so that \( C = \hat{g}(\mathbb{R}) \).
Corollary. Let $\Omega \subset \mathbb{R}^n$ be a simply connected region, and $\omega$ a 1-form on $\Omega$. If $\omega$ is closed, $\omega$ is exact.

Proof. Suppose $C$ is a closed curve in $X$. Then $C = \gamma(\partial R)$, so $C = \partial S$ where $S$ is parametrized by $R$. We then have

$$\int_C \omega = \int_S \omega = \int_S d\omega = \int_S 0 = 0.$$  

(by hypothesis)

Green's (them) corollary

By our previous theorem, if $\int_C \omega = 0$ for every closed curve in $X$, $\omega = df$. $\square$

We now consider the example

$$\omega = \frac{-y}{x^2+y^2} \, dy + \frac{x}{x^2+y^2} \, dx$$
We have previously computed
\[ d\omega = \left( \frac{\partial}{\partial y} \frac{y}{x^2+y^2} + \frac{\partial}{\partial x} \frac{x}{x^2+y^2} \right) \, dx \, dy \]
\[ = \left( \frac{x^2+y^2}{(x^2+y^2)^2} - 2y^2 + \frac{(x^2+y^2)^2 - 2x^2}{(x^2+y^2)^2} \right) \, dx \, dy \]
\[ = 0. \]

Now if \( \mathbf{g}(t) = \begin{bmatrix} r \cos t \\ r \sin t \end{bmatrix} \), we have
\[ \mathbf{g}^* \omega = \frac{-r \sin t}{r^2 \cos^2 t + r^2 \sin^2 t} \left( -r \sin t \right) \, dt \]
\[ + \frac{r \cos t}{r^2 \cos^2 t + r^2 \sin^2 t} \left( r \cos t \right) \, dt \]
\[ = dt \]
(for any \( r > 0 \)). Thus if \( C \) is any circle around the origin, we have
\[ \oint_{C} \omega = \int_{0}^{2\pi} dt = 2\pi. \]

Now suppose \( C \) is any simple closed curve which bounds a region including the origin.

There is a circle \( C' \) around the origin inside \( C \), and a region \( S \) so \( 2S = C - C' \).
We then know
\[ \int_{C} w = \int_{C'} w + \int_{S} dw = 2\pi + 0. \]

**Definition.** If \( C \) is a closed curve in \( \mathbb{R}^2 - \{0,0\} \), the winding number of \( C \) around \( 0 \) is given by

\[ \frac{1}{2\pi} \int_{C} \left( -\frac{y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy \right) \]

**Theorem.** The winding number is an integer. If \( C \) and \( C' \) can be deformed continuously to one another, their winding numbers are equal.