On the Chord Length to Arc Length Ratio for Open Curves Undergoing a Length-Rescaled Curvature Flow

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1 The Length-Rescaled Curvature Flow

We begin by defining a few properties of the new flow. Let ϕ be the curve parameter and t be a "time" parameter, so that curves undergoing the flow can be described by parametrizations $X(\phi, t)$. Let L be the total length of the curve. Then the evolution equation is given by:

$$X_t = \kappa N + \left(\frac{1}{L} \int \kappa^2 ds \right) X \tag{1}$$

Under this flow, we now have that

$$L_t = 0 \tag{2}$$

which is a condition that arises from integration and does not hold pointwise. We've also got that:

$$A_t = -2\pi - \left(\frac{1}{L} \int \kappa^2 ds \right) \int \langle X, N \rangle ds \tag{3}$$

where N is the unit normal vector, and:

$$\kappa_t = \frac{1}{\left| \frac{\partial X}{\partial \phi} \right|} \frac{\partial}{\partial \phi} \left(\frac{\frac{\partial \kappa}{\partial \phi}}{\left| \frac{\partial X}{\partial \phi} \right|} \right) + \kappa^3 - \frac{\kappa}{L} \int \kappa^2 ds \tag{4}$$

2 Theorem

Let d and l denote the chord length and arc length respectively between any two points p and q on an open curve, so that

$$d = |X(p,t) - X(q,t)| \tag{5}$$

$$l = \int_p^q \left| \frac{\partial X}{\partial \phi} \right| d\phi \tag{6}$$

We now prove the following:

Theorem: Let $X : \Gamma \times [0,T] \to \mathbf{R^2}$ *be an embedded solution of the length-rescaled curvature flow, where* $\Gamma \neq S^1$ *so that* l *is smoothly defined on* $\Gamma \times \Gamma$*. Then the minimum of* d/l *on* Γ *is nondecreasing; it is strictly increasing unless* $d/l \equiv 1$ *and* Γ *is a straight line segment.*

Proof. d and l are smooth functions off the diagonal, so it suffices to show that whenever their ratio attains a spatial minimum for some pair of points $(p,q) \in \Gamma \times \Gamma$ at some time $t_0 \in [0,T]$, we have that

$$\frac{d}{dt}\left(\frac{d}{l}\right)(p,q,t_0) \geq 0 \tag{7}$$

Assume without loss of generality that $p \neq q$ and that $s(p) \geq s(q)$ at t_0. Then by assumption we have that, following Huisken's notation, the first and second "variations" obey:

$$\delta(\xi)\left(\frac{d}{l}\right) = 0 \tag{8}$$

$$\delta^2(\xi)\left(\frac{d}{l}\right) \geq 0 \tag{9}$$

for variations $\xi \in T_p\Gamma_{t_0} \oplus T_q\Gamma_{t_0}$.

It will be helpful to reparametrize the curve locally around p and q using arc-length parameters u and v respectively, so that the curve is described near these points by $X(u,t_0)$ and $X(v,t_0)$. Then we'll need to define several vectors before continuing. Have e_1 and e_2 denote the unit tangent vectors along the curve at p and q respectively:

$$e_1 = \frac{\partial X(u,t_0)}{\partial u} \tag{10}$$

$$e_2 = \frac{\partial X(v,t_0)}{\partial v} \tag{11}$$

Let ω denote the unit vector in the direction from p to q :

$$\omega = \frac{X(v,t) - X(u,t)}{d} \tag{12}$$

Then note that our first variation obeys a Leibniz rule:

$$\delta\left(\frac{d}{l}\right) = \frac{\delta(d)}{l} - \frac{d}{l^2}\delta(l) \tag{13}$$

so that we need only compute the variations of d and l individually. In order to compute the first variation of d, we'll need to first compute d_u, d_v, and d_t. We pause to do that now (let the curvature vector $\vec{\kappa}$ denote $\kappa\mathbf{N}$ in the calculations to follow):

$$d_u = \frac{\langle X(u,t) - X(v,t), e_1\rangle}{d} = -\langle \omega, e_1\rangle \tag{14}$$

$$d_v = \frac{\langle X(u,t) - X(v,t), -e_2\rangle}{d} = \langle \omega, e_2\rangle \tag{15}$$

$$d_t = \tfrac{1}{d}\left\langle \left(X(u,t) - X(v,t)\right), \vec{\kappa}(u,t) + \left(\tfrac{1}{L}\int \kappa^2 ds\right) X(u,t) - \vec{\kappa}(v,t) - \left(\tfrac{1}{L}\int \kappa^2 ds\right) X(v,t)\right\rangle$$

$$= \langle -\omega, \vec{\kappa}(u,t) - \vec{\kappa}(v,t)\rangle - \left(\tfrac{1}{L}\int \kappa^2 ds\right)\langle \omega, X(u,t) - X(v,t)\rangle$$

$$= \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle + \frac{d}{L}\int \kappa^2 ds \tag{16}$$

We now consider the vanishing of the first variation along e_1 and e_2. We first compute the first variations of d and l in these directions and then plug them into the product rule for the ratio d/l :

$$\delta(e_1 \oplus 0)(d) = D_{e_1} d = \langle e_1, \nabla d\rangle = d_u = -\langle \omega, e_1\rangle \tag{17}$$

$$\delta(e_1 \oplus 0)(l) = -1 \tag{18}$$

$$\delta(0 \oplus e_2)(d) = D_{e_2} d = \langle e_2, \nabla d\rangle = d_v = \langle \omega, e_2\rangle \tag{19}$$

$$\delta(0 \oplus e_2)(l) = 1 \tag{20}$$

so, plugging these into equation (13) we've got

$$\delta(e_1 \oplus 0)\left(\frac{d}{l}\right) = \frac{d}{l^2} - \frac{\langle \omega, e_1\rangle}{l} = 0 \tag{21}$$

$$\delta(0 \oplus e_2)\left(\frac{d}{l}\right) = \frac{\langle \omega, e_2\rangle}{l} - \frac{d}{l^2} = 0 \tag{22}$$

from which it follows that

$$\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{l} \tag{23}$$

which we'll need to keep in mind for future calculations.

Now we turn to the second variation, for which we can write:

$$\delta^2 \left(\frac{d}{l} \right) = \frac{\delta^2(d)}{l} - 2\frac{\delta(d)\delta(l)}{l^2} + 2\frac{d(\delta(l))^2}{l^3} - \frac{d}{l^2}\delta^2(l) \geq 0 \tag{24}$$

We will consider two cases here:

Case 1: $e_1 = e_2$

All variations of l now vanish so we need only consider the first term in equation (24). Thus, we need to compute the second variation of d :

$$\delta^2(e_1 \oplus e_2)(d) = \left\langle H(d) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = d_{uu} + 2d_{uv} + d_{vv} \tag{25}$$

Now, using the relations we derived in equation (23), we can compute these second partials, for which it should turn out that:

$$d_{uu} = \frac{1}{d} - \frac{d}{l^2} - \langle \omega, \vec{\kappa}(u,t) \rangle \tag{26}$$

$$d_{vv} = \frac{1}{d} - \frac{d}{l^2} + \langle \omega, \vec{\kappa}(v,t) \rangle \tag{27}$$

$$d_{uv} = \frac{d}{l^2} - \frac{1}{d} \tag{28}$$

Plugging these into (25), we we get that

$$\delta^2(e_1 \oplus e_2) \left(\frac{d}{l} \right) = \frac{1}{l} \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \geq 0 \tag{29}$$

Case 2: $e_1 \neq e_2$

We'll now choose $\xi = e_1 \ominus e_2$ so that

$$\delta(e_1 \ominus e_2)(l) = -2 \tag{30}$$

and

$$\delta(e_1 \ominus e_2)(d) = -\langle \omega, e_1 + e_2 \rangle \tag{31}$$

We perform calculations exactly analogous to those in Case 1, computing the new partials and the new second variation of d. We end up being able to conclude the same inequality that we got for Case 1 in equation (29). See the separate write-up "Case 2 for Theorem 1" to see these calculations in detail.

We are now ready to turn our attention to a quantity of greater interest: the time derivative of the ratio d/l. With what we already know, we can go ahead and write:

$$\left(\frac{d}{l}\right)_t = \frac{d_t}{l} - \frac{d}{l^2}l_t = \frac{1}{l}\left\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\right\rangle + \frac{d}{l}\frac{\int \kappa^2 ds}{L} - \frac{d}{l^2}l_t \tag{32}$$

In order to say more about this expression we'll need to come up with an expression for l_t. Though it may seem that since this is a length-rescaled flow that particular term should vanish, we need to keep in mind that while the *total* length of the curve cannot change, we don't know what is happening locally, and l refers only to the arc length between two particular points p and q on the curve.

To begin with, recall that we can express l using equation (6), so that we can then write

$$l_t = \int_p^q \left|\frac{\partial X}{\partial \phi}\right|_t d\phi \tag{33}$$

Then we must find an expression for the integrand here:

$$\begin{aligned}
\left|\frac{\partial X}{\partial \phi}\right|_t &= \frac{\left\langle \frac{\partial X}{\partial \phi}, \frac{\partial^2 X}{\partial t \partial \phi}\right\rangle}{\left|\frac{\partial X}{\partial \phi}\right|} = \frac{\left\langle \frac{\partial X}{\partial \phi}, \frac{\partial^2 X}{\partial \phi \partial t}\right\rangle}{\left|\frac{\partial X}{\partial \phi}\right|} = \left\langle T, \frac{\partial}{\partial \phi}\left(\kappa N + \left(\tfrac{1}{L}\int \kappa^2 ds\right) X\right)\right\rangle \\
&= \left\langle T, \frac{\partial \kappa}{\partial \phi} N - \left|\frac{\partial X}{\partial \phi}\right| \kappa^2 T + \left(\frac{\left|\frac{\partial X}{\partial \phi}\right|}{L}\int \kappa^2 ds\right) T\right\rangle \\
&= -\kappa^2 \left|\frac{\partial X}{\partial \phi}\right| + \frac{1}{L}\left|\frac{\partial X}{\partial \phi}\right| \int \kappa^2 ds
\end{aligned} \tag{34}$$

So, integrating (34) with respect to ϕ from p to q, we'll get that

$$l_t = l\frac{\int \kappa^2 ds}{L} - \int_p^q \kappa^2 ds \tag{35}$$

With this expression in hand, we return to equation (32) and write:

$$\begin{aligned}
\left(\tfrac{d}{l}\right)_t &= \tfrac{1}{l}\left\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\right\rangle + \tfrac{d}{l}\tfrac{\int \kappa^2 ds}{L} + \tfrac{d}{l^2}\int_p^q \kappa^2 ds - \tfrac{d}{l}\tfrac{\int \kappa^2 ds}{L} \\
&= \tfrac{1}{l}\left\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\right\rangle + \tfrac{d}{l^2}\int_p^q \kappa^2 ds \geq \tfrac{d}{l^2}\int_p^q \kappa^2 ds
\end{aligned}$$

This last term is obviously greater than zero except on the diagonal of $\Gamma \times \Gamma$, where it is equal to zero. Thus:

$$\left(\frac{d}{l}\right)_t \geq \frac{d}{l^2}\int_p^q \kappa^2 ds \geq 0 \qquad (36)$$

and the theorem has been proven.