On a Chord Length to Arc Length Ratio for Closed Curves Undergoing a Length-Rescaled Curvature Flow

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We use the same notation as in the proof of the theorem for open curves. However, now we are going to want to define a new quantity since l is not smoothly defined for closed curves. Let $\psi : S^1 \times S^1 \times [0,T] \to \mathbf{R}$ be given by

$$\psi := \frac{L}{\pi} \sin\left(\frac{l\pi}{L}\right) \tag{1}$$

where l is the arc length between two points on the curve and L is the total length of that curve.

1 Theorem

Theorem: Let $X : S^1 \times [0,T] \to \mathbf{R^2}$ be a smooth embedded solution of the length-rescaled curve shortening flow. Then the minimum of $\frac{d}{\psi}$ is nondecreasing; it is strictly increasing unless $\frac{d}{\psi} \equiv 1$ and $X(S^1)$ is a round circle.

2 Proof

As in the first write-up, it suffices here to show that whenever $\frac{d}{\psi}$ attains a spatial minimum for some pair of points $(p,q) \in S^1 \times S^1$ at some time $t_0 \in [0,T]$, then

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p,q,t_0) \geq 0 \tag{2}$$

Let s be the arclength parameter at t_0 again, and $0 \leq s(p) \leq s(q) \leq \frac{1}{2}L(t_0)$, so that $l(p,q,t_0) = s(q) - s(p)$.

Let's again use Huisken's "variational" methods that we used in the proof of the theorem for open curves. So, by assumption we have that:

$$\delta(\xi)\left(\frac{d}{\psi}\right)(p,q,t_0)=0, \tag{3}$$

$$\delta^2(\xi)\left(\frac{d}{\psi}\right)(p,q,t_0)\geq 0 \tag{4}$$

for variations $\xi \in T_pS^1_{t_0} \oplus T_qS^1_{t_0}$.

Let's first consider, just as we did before, the variations $\xi = e_1 \oplus 0$ and $\xi = 0 \oplus e_2$. Because our definitions of d and l have not changed we can use the previous computations of the derivatives and variations here in order to say that

$$\delta(e_1\oplus 0)(d) = -\langle\omega, e_1\rangle \tag{5}$$

and

$$\delta(0\oplus e_2)(d) = \langle\omega, e_2\rangle \tag{6}$$

$$\delta(e_1\oplus 0)(l) = -1 \tag{7}$$

$$\delta(0\oplus e_2)(l) = 1 \tag{8}$$

Then, we can also compute

$$\delta(e_1\oplus 0)(\psi) = \frac{d}{dl}(\psi)\delta(e_1\oplus 0)(l) = -\cos\left(\frac{l\pi}{L}\right) \tag{9}$$

$$\delta(0\oplus e_2)(\psi) = \frac{d}{dl}(\psi)\delta(0\oplus e_2)(l) = \cos\left(\frac{l\pi}{L}\right) \tag{10}$$

Then, remembering that $\delta(\frac{d}{\psi}) = \frac{\delta(d)}{\psi} - \frac{d}{\psi^2}\delta(\psi)$, we can plug in equations (5), (6), (9), and (10) to show that

$$\delta(e_1\oplus 0)\left(\frac{d}{\psi}\right) = \frac{-\langle\omega, e_1\rangle}{\psi} + \frac{d}{\psi^2}\delta(\psi) = 0 \tag{11}$$

and

$$\delta(0\oplus e_2)\left(\frac{d}{\psi}\right) = \frac{\langle\omega, e_2\rangle}{\psi} - \frac{d}{\psi^2}\delta(\psi) = 0 \tag{12}$$

from which it follows that

$$\langle\omega, e_1\rangle = \langle\omega, e_2\rangle = \frac{d}{\psi}\cos(\frac{l\pi}{L}) \tag{13}$$

which we'll want to keep in mind.

Now we consider the second variation, which satisfies:

$$\delta^2\left(\frac{d}{\psi}\right) = \frac{\delta^2(d)}{\psi} - 2\frac{\delta(d)\delta(\psi)}{\psi^2} + 2\frac{d(\delta(\psi))^2}{\psi^3} - \frac{d}{\psi^2}\delta^2(\psi) \geq 0 \tag{14}$$

Once more, we'll consider two cases.

CASE 1: $e_1 = e_2$

Choose $\xi = e_1 \oplus e_2$. Because variations of l vanish in this case, all variations of ψ will also vanish since, as we've seen, variations of l pop out upon differentiation of ψ. Thus, we have reduced the problem to computing $\delta^2(e_1 \oplus e_2)\left(\frac{d}{\psi}\right) = \frac{\delta^2(d)}{\psi}$. We computed the numerator in the proof of the first theorem, so we have that:

$$\frac{1}{\psi}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle \geq 0 \tag{15}$$

CASE 2: $e_1 \neq e_2$

Choose $\xi = e_1 \ominus e_2$. Variations of l no longer vanish; now $\delta(l) = -2$. So now from equation (14) we have that $\delta^2\left(\frac{d}{\psi}\right) = \frac{\delta^2(d)}{\psi} - 2\frac{\delta(d)\delta(\psi)}{\psi^2} + 2\frac{d(\delta(\psi))^2}{\psi^3} \geq 0$.

First we'll compute $\delta^2(d)$ for this variation. First recall that:

$$\delta^2(e_1 \ominus e_2)(d) = \left\langle H(d)\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\rangle = d_{uu} - 2d_{uv} + d_{vv} \tag{16}$$

We'll pause to compute these second partials, which are now different than in Case 1. Begin with

$$d_u = -\langle \omega, e_1\rangle \tag{17}$$

$$d_v = \langle \omega, e_2\rangle \tag{18}$$

To begin computing the second partials of each of these, we'll need to recall equation (13) and make appropriate substitutions. We should get that:

$$d_{uu} = \frac{1}{d} - \frac{d}{\psi^2}\cos^2(\frac{l\pi}{L}) - \langle \omega, \vec{\kappa}(u,t)\rangle \tag{19}$$

$$d_{vv} = \frac{1}{d} - \frac{d}{\psi^2}\cos^2(\frac{l\pi}{L}) + \langle \omega, \vec{\kappa}(v,t)\rangle \tag{20}$$

$$d_{uv} = \frac{d}{\psi^2}\cos^2(\frac{l\pi}{L}) - \frac{1}{d}\langle e_1, e_2\rangle \tag{21}$$

We plug these into equation (16) and use the fact that $2\langle e_1, e_2\rangle = |e_1 + e_2|^2 - 2$ in order to get:

$$\delta^2(e_1 \ominus e_2)(d) = \frac{1}{d}|e_1 + e_2|^2 - 4\frac{d}{\psi^2}\cos^2(\frac{l\pi}{L}) + \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle \tag{22}$$

Now, in this case, since $\omega \parallel (e_1 + e_2)$, we can rewrite the first term in equation (22) as $\frac{1}{d}\langle \omega, e_1 + e_2\rangle^2$, and then use equation (13) so that (22) simplifies nicely to

$$\delta^2(e_1 \ominus e_2)(d) = \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \tag{23}$$

So we return to (14) and plug this in:

$$\delta^2 \left(\frac{d}{\psi} \right) = \frac{1}{\psi} \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle - 2\frac{\delta(d)\delta(\psi)}{\psi^2} + 2\frac{d(\delta(\psi))^2}{\psi^3} \geq 0 \tag{24}$$

Now, since

$$\delta(e_1 \ominus e_2)(d) = -\langle \omega, e_1 + e_2 \rangle \tag{25}$$

and

$$\delta(e_1 \ominus e_2)(l) = -2 \tag{26}$$

we can go ahead and plug these into (24). We should get:

$$\delta^2 \left(\frac{d}{\psi} \right) = \frac{1}{\psi} \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle - \frac{4}{\psi^2} \langle \omega, e_1 + e_2 \rangle \cos(\frac{l\pi}{L}) + 8\frac{d}{\psi^3} \cos^2(\frac{l\pi}{L}) \geq 0 \tag{27}$$

but by simplifying the middle term using equation (13) we have the entire relation reducing to:

$$\delta^2 \left(\frac{d}{\psi} \right) = \frac{1}{\psi} \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \geq 0 \tag{28}$$

and we've got the same result as in Case 1.

We are now ready to turn to the time derivative of the ratio $\frac{d}{\psi}$. We can start by writing:

$$\left(\frac{d}{\psi} \right)_t = \frac{d_t}{\psi} - \frac{d}{\psi^2}\psi_t = \frac{1}{\psi} \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle + \frac{d}{L\psi} \left(\int \kappa^2 ds \right) - \frac{d}{\psi^2}\psi_t \tag{29}$$

We already had the time derivative of d from the proof of the first theorem, but now we'll need to come up with an expression for the time derivative of ψ. We'll also need to recall the expression for the time derivative of l from the first proof.

$\psi_t = \frac{L}{\pi} \cos(\frac{l\pi}{L})(\frac{\pi}{L})l_t =$

$$\psi_t = \cos(\frac{l\pi}{L}) \left(\frac{l}{L} \int \kappa^2 ds - \int_p^q \kappa^2 ds \right) \tag{30}$$

Now then we can rewrite (29) as:

$$\left(\frac{d}{\psi}\right)_t = \frac{1}{\psi}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle + \frac{d}{L\psi}\left(\int \kappa^2 ds\right) - \frac{d}{\psi^2}\cos(\tfrac{l\pi}{L})\left(\frac{l}{L}\int \kappa^2 ds - \int_p^q \kappa^2 ds\right)$$

$$\geq \frac{d}{L\psi}\left(\int \kappa^2 ds\right) + \frac{d}{\psi^2}\cos(\tfrac{l\pi}{L})\left(\int_p^q \kappa^2 ds - \frac{l}{L}\int \kappa^2 ds\right)$$

$$= \frac{dl}{\psi^2}\left(\frac{\psi}{l}\frac{\int \kappa^2 ds}{L} + \frac{1}{l}\cos(\tfrac{l\pi}{L})\int_p^q \kappa^2 ds - \cos(\tfrac{l\pi}{L})\frac{\int \kappa^2 ds}{L}\right)$$

Now we are interested in saying something about the sign of this expression, so we can ignore the $\frac{dl}{\psi^2}$, which is positive, and try to say something about what's inside the parentheses.

$$\frac{\psi}{l}\frac{\int \kappa^2 ds}{L} + \frac{1}{l}\cos(\tfrac{l\pi}{L})\int_p^q \kappa^2 ds - \cos(\tfrac{l\pi}{L})\frac{\int \kappa^2 ds}{L}$$

$$= \left(\frac{\psi}{l} - \cos(\tfrac{l\pi}{L})\right)\frac{\int \kappa^2 ds}{L} + \frac{1}{l}\cos(\tfrac{l\pi}{L})\int_p^q \kappa^2 ds$$

Although until now we've done things in some generality with an L term, we can recall now that we initially stated the theorem for curves of length 2π. Furthermore, it is useful to return to the explicit definition of ψ. Going ahead and plugging in an $\frac{L}{\pi}\sin(\frac{l\pi}{L})$ where we see a ψ and a 2π wherever we see an L we get:

$$\left(\frac{2}{l}\sin\left(\frac{l}{2}\right) - \cos\left(\frac{l}{2}\right)\right)\frac{\int \kappa^2 ds}{L} + \frac{1}{l}\cos\left(\frac{l}{2}\right)\int_p^q \kappa^2 ds \qquad (31)$$

Returning once again to initial assumptions, we had said that $l(p,q,t_0) < \frac{L_{t_0}}{2}$. So $l < \pi$ and we know that the rightmost term in (31) is positive. We now show that the leftmost term in the parentheses is positive as well. Call $x = \frac{l}{2}$. Then that expression in the parentheses becomes

$$\frac{1}{x}\sin(x) - \cos(x) = \cos(x)\left(\frac{1}{x}\tan(x) - 1\right) = \frac{1}{x}\cos(x)\left(\tan(x) - x\right).$$

For the same reason that we could say the rightmost term in (31) was positive, we can now say that $\frac{1}{x}\cos(x)$ is positive. So we are left with deciding whether or not $(\tan(x) - x)$ is positive. That this is true between 0 and $\frac{\pi}{2}$ is easily verifiable. For example, one can simply examine the Taylor expansion for the range in which we are interested:

For $|x| < \frac{\pi}{2}$:

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + ... \qquad (32)$$

Thus, we've shown that each of the terms in (31) is positive, and so $\left(\frac{d}{\psi}\right)_t \geq 0$, as desired. This completes the proof of the theorem.