Geodesics and Abstract Geometries

We have given the geodesic equations in one form (natural from the point of view of geodesic curvature).

Theorem. \( \alpha(s) = x(u(s), v(s)) \) a geodesic \( \iff \)

\[
\frac{d}{ds} (Eu' + Fv') = \frac{1}{a} \left( E_u(u')^2 + 2F_u u' v' + G_u(v')^2 \right)
\]

\[
\frac{d}{ds} (F u' + G v') = \frac{1}{a} \left( E_v(u')^2 + 2F_v u' v' + G_v(v')^2 \right)
\]

and \( E(u')^2 + F u' v' + G(v')^2 = 1 \).

Proof. Observe

\[
\frac{d}{ds} (Eu' + Fv') = (E_u u' + E_v v') u' + E_u''
\]

so we have

\[
+ (F_u u' + F_v v') v' + F_v''
\]

\[
\frac{d}{ds} (Eu' + Fv') - Eu'' - Fv'' = (E_u u' + E_v v') u'
\]

\[
+ (F_u u' + F_v v') v'
\]
adding this to the first geodesic equation
\[ u'' E + v'' F = - \frac{1}{a} E_u (u')^2 - E_v u' v' - (F_v - \frac{1}{a} G_u) v' \]
yields
\[ \frac{d}{ds} (E u' + F v') = \frac{1}{a} E_u (u')^2 + F_u u' v' + \frac{1}{a} G_u (v')^2. \]
The other equation is similar.
The third enforces that \( \alpha \) is arclength parametrized. \( \square \)

Since everything is expressed in terms of \( E, F, G \), do we still need \( X \)?

**Definition.** A **metric** on \( U \subset \mathbb{R}^2 \) is a smooth function \( I : U \rightarrow \text{Mat}_{2 \times 2}(\mathbb{R}) \) so that
\[
I(p) = \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix}
\]
is a symmetric positive definite matrix for \( p \in U \).
Every parametrization $X: U \to \mathbb{R}^3$ induces a metric, because

$$I_p = (DX(\tilde{\rho}))^T (DX(\tilde{\rho}))$$

is s.p.d. by construction.

Example. The upper half-plane with

$$E = G = \frac{1}{v^2}, \quad F = 0.$$  

now $E_u = G_u$ so the equations become

$$\frac{d}{ds} \left( \frac{u^1}{v^2} \right) = 0$$

$$\frac{d}{ds} \left( \frac{v^1}{v^2} \right) = \frac{1}{v^2} \left( - \frac{2}{v^3} (u^1)^2 - \frac{2}{v^3} (v^1)^2 \right)$$

$$\frac{1}{v^2} (u^1)^2 + \frac{1}{v^2} (v^1)^2 = 1$$
Now
\[ \frac{d}{ds} \left( \frac{u'}{v^2} \right) = 0 \]
so there is a constant \( C_2 \) so that
\[ u' = C_2 v^2. \]
If \( C_2 = 0 \), then the equations become
\[ u(s) = \text{constant} \]
\[ \frac{d}{ds} \left( \frac{v'}{v^2} \right) = -\frac{(v')^2}{v^3} \]
\[ \frac{(v')^2}{v^2} = 1 \]
The third equation is
\[ v' = \pm v, \text{ which has } v(s) = ce^{\pm \gamma} \]
Now if \( v(s) = ce^{-s} \), then

\[
v'(s) = -ce^{-s}
\]

\[
\frac{v'(s)}{v^2(s)} = \frac{-ce^{-s}}{c^2e^{-2s}} = -\frac{1}{c}e^s
\]

and

\[
\frac{d}{ds} \frac{v'(s)}{v^2(s)} = -\frac{1}{c}e^s
\]

while

\[
-\frac{(v')^2}{v^3} = -\frac{c^2e^{-2s}}{c^3e^{-3s}} = -\frac{1}{c}e^s
\]

On the other hand, if \( v(s) = ce^s \),

\( v'(s) = ce^s \), and we have

\[
\frac{v'(s)}{v^2(s)} = \frac{ce^s}{c^2e^{2s}} = \frac{1}{c}e^{-s}
\]
so
\[
\frac{d}{ds} \left( \frac{v^1}{v^2} \right) = -\frac{1}{c} \, e^{-s}
\]

while
\[
-\frac{(v^1)^2}{v^3} = -\frac{c^2 e^{2s}}{c^3 e^{3s}} = -\frac{1}{c} \, e^{-s}.
\]

We conclude that
\[
u(s) = c_0, \quad v(s) = c_1 e^{\pm s}
\]
is always a geodesic.

Now suppose that \( c_2 \neq 0 \) and
\[
u^1 = c_2 v^2.
\]
Then
\[
\frac{(v^1)^2}{v^2} + \frac{(v^1)^2}{v^2} = 1 \implies c_2^2 v^2 + \frac{(v^1)^2}{v^2} = 1.
\]
This second and third equation yield

\[
\frac{d}{ds} \left( \frac{v^1}{v^2} \right) = \frac{1}{2} \left( - \frac{2}{v^3} (u^1)^2 - \frac{2}{v^3} (v^1)^2 \right) \\
= - \frac{1}{v} \left( \frac{(u^1)^2}{v^2} + \frac{(v^1)^2}{v^2} \right)^{1/2} \\
= - \frac{1}{v}.
\]

Now \( \frac{d}{ds} \left( - \frac{1}{v} \right) = \frac{v^1}{v^2} \), so we can see that this equation is

\[
\frac{d^2}{ds^2} \left( - \frac{1}{v} \right) = - \frac{1}{v} \iff \frac{d^2}{ds^2} \left( \frac{1}{v} \right) = \frac{1}{v}
\]

We conclude that (in general)

\[
\frac{1}{v} = C_o \cosh (s + \zeta_0)
\]

so

\[
v = C_o \text{sech} (s + \zeta_1), \text{ where } C_o > 0.
\]
and
\[ v'(s) = c_0 \text{sech}(s+c_1) \tanh(s+c_1) \]
So
\[ \frac{v'}{v} = \tanh(s+c_1) \]
Returning to \[ u' = c_a v^2 \] and
\[ c_a^2 v^2 + \left( \frac{v'}{v} \right)^2 = 1 \]
we see
\[ c_a^2 c_0^2 \text{sech}^2(s+c_1) + \tanh^2(s+c_1) = 1 \]
Since \[ \text{sech}^2 x + \tanh^2 x = 1 \], we get \[ c_a^2 c_0^2 = 1 \], so we conclude
\[ u'(s) = \frac{1}{c_0} \cdot c_0^2 \text{sech}^2(s + c_4) \]

and

\[ u(s) = c_0 \tanh(s + c_4) + c_2. \]

Now we just need to understand \( c_0, c_4, \) and \( c_2. \) Notice that

\[ (u - c_2)^2 + v^2 = c_0^2 \tanh^2(s + c_4) + c_0^2 \text{sech}^2(s + c_4) = c_0^2. \]

Thus the solutions are really nice!
We call this the hyperbolic plane.

Triangles in the hyperbolic plane are interesting:

Angles sum to less than $\pi$.

A triangle with angle sum zero!
Definition. If $F: \mathbb{R}^2 \to \mathbb{R}^2$, and $I_p$ is a metric on $\mathbb{R}^2$, we say $F$ is an isometry of $I_p$ if

$$(DF(\hat{p}))^T I_{F(\hat{p})} DF(\hat{p}) = I_{\hat{p}}$$

Example. If $F: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $F(\hat{x}) = A\hat{x} + \hat{c}$ where $A$ is an orthogonal matrix and $I_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$DF(\hat{p})^T I_{DF(\hat{p})} DF(\hat{p}) = A^T I A = I.$$ So translations and rotations (and reflections) are Euclidean isometries.

What about hyperbolic isometries?
Definition. If \( \hat{z} \in \mathbb{R}^2 \) and \( r \in \mathbb{R}^+ \), the map \( R(\hat{p}) = \frac{r^2}{\lambda^2} (\hat{z} - \hat{p}) + \hat{z} \), where \( \lambda = ||\hat{p} - \hat{z}|| \) is called inversion.

Lemma. \( DR(\hat{p}) = \frac{r^2}{\lambda^2} A \), where \( A = (I - 2\hat{p}\hat{p}^T) \) is reflection over the direction orthogonal to \( \hat{p} - \hat{z} \), so \( A \) is an orthogonal matrix.
Proposition. \( R \) is an isometry of \( \mathbb{R}^2 \) with the hyperbolic metric
\[
I_p = \begin{bmatrix}
\frac{1}{v^2} & 0 \\
0 & 1/v^2
\end{bmatrix}, \text{ where } \hat{p} = [u, v].
\]

Proof. Notice that \( I_p = \frac{1}{v^2} I_2 \), and the \( u \)-coordinate of \( R(\hat{p}) \) is
\[
\langle R(\hat{p}), \hat{e}_2 \rangle = \langle \frac{r^2}{\lambda^2} (\hat{p} - \hat{z}) + \hat{z}, \hat{e}_2 \rangle
= \frac{r^2}{\lambda^2} \langle \hat{p}, \hat{e}_2 \rangle = \frac{r^2}{\lambda^2} \cdot v
\]
Therefore, \( I_{R(\hat{p})} = \frac{\lambda^4}{r^4 v^2} \), so
\[
DR(\hat{p})^T I_{R(\hat{p})} DR(\hat{p}) = \frac{r^2}{\lambda^2} A^T \frac{\lambda^4}{r^4 v^2} I A \frac{r^2}{\lambda^2}
= \frac{1}{v^2} A^T A = \frac{1}{v^2} I
= I_{\hat{p}}. \quad \square
These isometries tell us amazing things about the hyperbolic plane.

**Congruent right triangles**

\[ \triangle abc \cong \triangle dbc \]