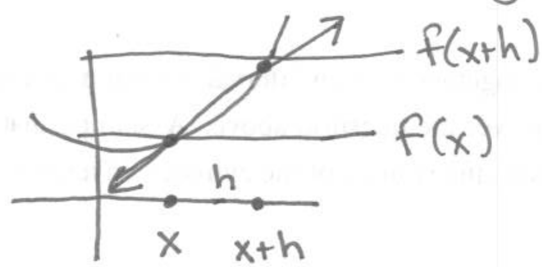
Robot Calculus – Lecture 1. (3.1, HWT)

Let $f(x)$ be a function. Our big idea is to measure the change in f as x changes.



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{x+h - x}$$

Example. $f(t)$ = miles from Atlanta. t = # of hours since start

At noon, you start in Atlanta. $f(0) = 0$. At 1 pm, you arrive in Athens, $f(1) = 75$.

Average rate of change vs. speedometer (instant. rate of change)

Definition. The derivative $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Limits work just as you hope except

$$\lim_{x \to a} \frac{f(x)}{g(x)} \text{ is unknown if } \lim_{x \to a} g(x) = 0.$$

Basically every function in this class is continuous, which means $\lim\limits_{x \to a} f(x) = f(a)$, ~~as long~~ when $f(a)$ exists.

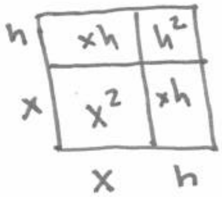
$$\text{position} \xrightarrow{d/dt} \text{velocity} \xrightarrow{d/dt} \text{acceleration}$$

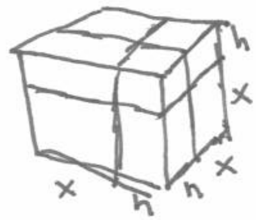
~~Finding the~~

Robot Calculus - Lecture 2. (3.2, HWT)

Consider $f(x) = x$. $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{(x+h) - x}{(x+h) - x} = \lim_{h \to 0} 1 = 1.$

Now do $f(x) = x^2$. $\frac{d}{dx} f(x) = 2x$.





$= x^3 + 3x^2h + 3xh^2 + h^3.$

$$(x^3 + 3x^2h + 3xh^2 + h^3)(x+h) = x^4 + 3x^3h + 3x^2h^2 + xh^3 + x^3h + 3x^2h^2 + 3xh^3 + h^4$$

$$= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

We can turn this into a game!

$1x + 1h$

$1x^2 + 2xh + 1h^2$ = Pascal's triangle.

$1x^3 \quad 3x^2h \quad 3xh^2 + 1h^3$

1 4 6 4 1

1 5 10 10 5 1

So $(x+h)^{17} = x^{17} + 17x^{16}h + \_\, x^{15}h^2 + \dots + h^{17}$,

and

$$\frac{d}{dx} x^{17} = \lim_{h \to 0} \frac{(x+h)^{17} - x^{17}}{h} = \lim_{h \to 0} \frac{17x^{16}h + \_\, x^{15}h^2 + \dots + h^{17}}{h}$$

$$= \lim_{h \to 0} 17x^{16} + \_\, x^{15}h + \dots + h^{16} = 17x^{16}.$$

Rule: $\frac{d}{dx} x^n = n x^{n-1}$. Sum rule, constant rule, derivative of constant fn = 0.

Robot Calculus - Lecture 3. (3.3, HWT)

constant

Newton's laws. $f = ma$. force of gravity $= \cancel{K}\dfrac{m_1 m_2}{d^2} \approx g m_1$.

So a thrown object near earth obeys

$$g\cancel{m} = \cancel{m}a, \quad a = g \approx -9.8\ m/s^2 \text{ or } -32\ ft/sec^2$$

regardless of mass, in the y direction, and $0 = ma \Rightarrow a =$ in the x-direction (no force, no acceleration!).

$$\text{position } p(t) \xrightarrow{d/dt} \text{velocity } v(t) \xrightarrow{d/dt} \text{acceleration } a(t)$$

If

$$p(t) = At^2 + Bt + C, \quad v(t) = 2At + B, \quad a(t) = 2A.$$

So for <u>vertical</u> motion, $p(t) = y(t)$, $a_y(t) = g$,

$$v_y(t) = gt + v_0 \leftarrow \text{velocity at time 0}$$
$$y(t) = \tfrac{g}{2}t^2 + v_0 t + y_0 \leftarrow \text{position at time 0}$$

For <u>horizontal</u> motion, $a_x(t) = 0$, so we have

$$v_x(t) = V_0, \quad x(t) = V_0 t + X_0 \leftarrow \text{position at time 0}.$$

Example. At time 0, a tennis ball is thrown with an x-velocity of 2 m/s and a y-velocity of 4 m/s from the point (0, 2). What path does it follow? Where does it land?

Lab. Fitting, predicting, and intercepting!

Robot Calculus - Lecture 4 (3.2 HWT)

$$\frac{d}{dx} f(x)g(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0} g(x)\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= f(x)g'(x) + f'(x)g(x).$$ (product rule).

Example. $(x^7+3x^5+x+7)(x^6+3x-6)$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

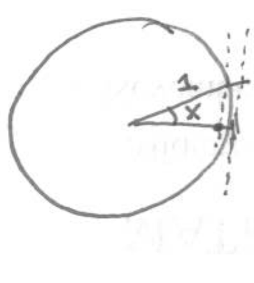
$$= \cancel{\lim}\ g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{g(x)g(x+h)} - f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h\,g(x)g(x+h)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$ (quotient rule)

Example. $\frac{x^2+3}{x-7}$.

In-class creativity exercise. $1 = \sin^2\theta + \cos^2\theta = \dots$.
(Find as many equivalent forms as you can in 5 minutes).

Limit of $\sin x/x$.



Area of inner triangle $= \frac{1}{2}\sin\theta\cos\theta$

Area of sector $= \pi\cdot\theta/2\pi = \frac{1}{2}\theta$.

Area of outer triangle $= \frac{1}{2}\cdot 1\cdot\tan\theta$.

$\Downarrow$

$$\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\sin\theta/\cos\theta$$

$\Downarrow$

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \Rightarrow \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1.$$

Robot Calculus - Lecture 5 (3.4 HWT)

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}(\sin x)\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\cos x\,\frac{\sin h}{h}$$

$$= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h}\cdot\frac{\cos h + 1}{\cos h + 1} + \cos x$$

$$= \sin x \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)} + \cos x = \sin x \lim_{h\to 0}\overset{1}{\frac{-\sin h}{h}}\;\overset{0/2}{\frac{\sin h}{\cos h + 1}} + \cos x$$

$$= \cos x.$$

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= -\sin x\cdot\lim_{h\to 0}\frac{\sin h}{h} + \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} = -\sin x.$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = (\text{quotient rule}) = \frac{1}{\cos^2 x} = \sec^2 x.$$

Circular motion. A fan blade spins at 500 rpm. What ~~is~~ are the vertical and horizontal position, velocity, and acceleration of a point 1 ft from the center? As a fn of θ? At $\theta = \pi/3$?

Parametrizing the circle by $\left(r\cos\frac{\theta}{K}, r\sin\frac{\theta}{K}\right)$. How to determine r and K.

Other trig derivatives.

Robot Calculus - Lecture 6.

Rates of change as we change the units of time, and as we multiply t by K. $\frac{d}{dt} f(Kt) = Kf'(Kt)$.

Chain Rule. $\frac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$.

what it means to compose functions. $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \xrightarrow{g(f(x))} \dots$

Algebra of composition, (examples).

Creativity exercise Write $\sin(x^2+1) + (x^2+1)^4$ as a compositi in as many ways as you can in 5 minutes.

and chain rule examples.

Circular motion examples for $(r\cos\theta/k, r\sin\theta/k)$.

$\boxed{\text{Target Tracking Lab!!!}}$

Robot Calculus - Lecture 6

Rates of change and changing units of time. $\frac{d}{dt} f(Kt)$.

composition of functions $\xrightarrow{x} \boxed{g} \xrightarrow{g(x)} \boxed{f} \xrightarrow{f(g(x))}$

algebra of compositions. examples.

Creativity exercise. Write $\sin(x^2+1) + (x^2+1)^4$ as a composition in as many ways as you can.

Chain Rule. $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$.

Examples. Derivatives of $(r \cos \theta/k, r \sin \theta/k)$.

Lab: Trajectory of stuff launched from a wheel.

Robot Calculus - Lecture 7. (3.3 HWT)

What about $f(x) = a^x$?

$$\frac{d}{dx} a^x = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = a^x \cdot \left(\lim_{h\to 0} \frac{a^h - 1}{h} \right)$$

where $\lim_{h\to 0} \frac{a^h-1}{h} = g(a)$ for some function g. So we have

$$\frac{d}{dx} a^x = a^x g(a).$$

That's weird. What's $g(a)$? Calculator experiments.

Definition. e is the number for which $g(e) = 1$, or $\frac{d}{dx} e^x = e^x$.

$e \approx 2.71828 1828 \ldots$

Recall that $\log_a x =$ the power y so that $a^y = x$.

Example. $\log_{10} x = \#$ of digits in x. $\log_e e^x = x$

$$\frac{d}{dx} e^{\ln x} = e^{\ln x} \cdot \frac{d}{dx} \ln x = x \cdot \frac{d}{dx} \ln x.$$

$$\frac{d}{dx} x = 1. \quad \text{So } x \cdot \frac{d}{dx} \ln x = 1 \text{ and } \frac{d}{dx} \ln x = 1/x.$$

Examples for $\ln x$ and e^x derivatives.

Challenge problem. Derivative of x^x. Rewrite as many ways as you can. What about $(x^x)^x$?