Definition. For a unit-speed curve $\vec{\alpha}(s)$ in $\mathbb{R}^3$ with $\vec{\alpha}''(s) \neq \vec{0}$, we say

$T(s) = \vec{\alpha}'(s)$ is called the tangent vector

$N(s) = \dfrac{\vec{\alpha}''(s)}{\|\vec{\alpha}''(s)\|}$ is called the normal vector

$B(s) = T(s) \times N(s)$ is called the binormal.

Definition. The Frenet frame is the map $F(s) = \begin{bmatrix} T(s) & N(s) & B(s) \end{bmatrix} : \mathbb{R} \to Mat_{3\times3}$.

Proposition. The Frenet frame $F(s) \in SO(3)$, or (equivalently) T, N, B are always an orthonormal basis for $\mathbb{R}^3$ so that $\det([T\ N\ B]) = +1$.

Proof. Since $\vec{\alpha}$ is arclength parametrized, $T(s) = \vec{\alpha}'(s)$ is unit length. So

$$\frac{d}{ds}\langle T(s), T(s)\rangle = 2\langle \vec{\alpha}''(s), \vec{\alpha}'(s)\rangle = 0$$

Thus

$$\langle N(s), T(s) \rangle = \left\langle \frac{\vec{\alpha}''(s)}{\|\vec{\alpha}''(s)\|}, \vec{\alpha}'(s) \right\rangle = 0.$$

and $T(s), N(s)$ are orthogonal unit vectors. Since $B(s) = T(s) \times N(s)$, $B(s)$ is orthogonal to $T(s)$ and $N(s)$.

$$\begin{aligned} \|B(s)\|^2 &= \|T(s) \times N(s)\|^2 \\ &= \|T(s)\|^2 \|N(s)\|^2 - \langle T(s), N(s) \rangle^2 \\ &= 1. \end{aligned}$$

Finally,

$$\begin{aligned} \det \begin{bmatrix} T & N & B \end{bmatrix} &= \langle T, N \times B \rangle \\ &= \langle B, T \times N \rangle \\ &= \langle T \times N, T \times N \rangle = 1. \quad \blacksquare \end{aligned}$$

Proposition. There are scalar functions $\kappa > 0$ and τ so that

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

We call these the **Frenet equations.**

Proof. Since T, N, B is an orthonormal basis for $\mathbb{R}^3$,

$$\begin{aligned} T' &= \langle T', T\rangle T + \langle T', N\rangle N + \langle T', B\rangle B \\ N' &= \langle N', T\rangle T + \langle N', N\rangle N + \langle N', B\rangle B \\ B' &= \langle B', T\rangle T + \langle B', N\rangle N + \langle B', B\rangle B \end{aligned}$$

Since $\langle T, T\rangle \equiv \langle N, N\rangle \equiv \langle B, B\rangle \equiv 1$, differentiating wrt s shows

$$2\langle T', T\rangle \equiv 2\langle N', N\rangle \equiv 2\langle B', B\rangle \equiv 0.$$

Similarly, since $\langle T, N\rangle = \langle N, B\rangle = \langle B, T\rangle \equiv 0$, we learn

$$\langle T', N\rangle + \langle T, N'\rangle = 0$$
$$\langle N', B\rangle + \langle N, B'\rangle = 0$$
$$\langle T', B\rangle + \langle T, B'\rangle = 0$$

Define $\kappa = \|\vec{\alpha}''\| > 0$. Then

$$\langle T', N\rangle = \left\langle \vec{\alpha}'', \frac{\vec{\alpha}''}{\|\vec{\alpha}''\|}\right\rangle = \frac{\langle \vec{\alpha}'', \vec{\alpha}''\rangle}{\|\vec{\alpha}''\|} = \|\vec{\alpha}''\| = \kappa$$

Define $\tau = \langle N', B\rangle$. Finally, observe that

$$\langle T', B\rangle = \langle \vec{\alpha}'', B\rangle = \langle \kappa N, B\rangle = 0.$$

This completes the proof. $\boxtimes$

<u>Definition</u>. We call $\kappa = \langle T', N\rangle$ and $\tau = \langle N', B\rangle$ the <u>curvature</u> and <u>torsion</u> of the curve $\vec{\alpha}(s)$.

Geometric interpretation.

If $\vec{\alpha}(s)$ is the path of a rollercoaster moving at constant speed which oriented so that the rider's vertical direction is $N(s)$ and forward is $T(s)$ then

$$\kappa(s) = \|\vec{\alpha}''(s)\| = \text{"vertical" acceleration experienced by rider}$$

$$\tau(s) = \langle N', B \rangle = \text{"twisting" experienced by rider.}$$

See Duran, Tsuzoki 2018, "Procedural method for finding roller coaster rails centerlines based on heart-line acceleration criteria."

As we saw in the video, κ and τ describe how the Frenet frame changes.

<u>Proposition</u>. If $S = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$ and

$F = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T & N & B \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$ is the Frenet frame,

the Frenet equations are $F' = FS$.

<u>Proof</u>. We just check

$$FS = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T & N & B \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \kappa N & -\kappa T + \tau B & -\tau B \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T' & N' & B' \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = F'.$$

$\blacksquare$

We are rarely in the case where $\vec{\alpha}(s)$ is arclength parametrized, so we prove:

<u>Proposition</u>. If $\vec{\alpha}(t)$ is a regular parametrized curve in $\mathbb{R}^3$ with $\vec{\alpha}''(t) \neq \vec{0}$,

$$T = \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}$$

$$N = \frac{(\vec{\alpha}' \times \vec{\alpha}'') \times \vec{\alpha}'}{\|\vec{\alpha}'\| \, \|\vec{\alpha}' \times \vec{\alpha}''\|} = \frac{\langle \vec{\alpha}', \vec{\alpha}' \rangle \vec{\alpha}'' - \langle \vec{\alpha}', \vec{\alpha}'' \rangle \vec{\alpha}'}{\|\vec{\alpha}'\| \, \|\vec{\alpha}' \times \vec{\alpha}''\|}$$

$$B = \frac{\vec{\alpha}' \times \vec{\alpha}''}{\|\vec{\alpha}' \times \vec{\alpha}''\|}$$

while

$$K = \frac{\|\vec{\alpha} \times \vec{\alpha}''\|}{\|\vec{\alpha}'\|^3} \quad \text{and} \quad \tau = \frac{\langle \vec{\alpha}', \vec{\alpha}'' \times \vec{\alpha}''' \rangle}{\|\vec{\alpha}' \times \vec{\alpha}''\|^2}$$

Proof. Recall that for any regular $\vec{\alpha}(t)$

$$s(t) = \int_0^t \|\vec{\alpha}'(x)\|\,dx, \quad s'(t) = \|\vec{\alpha}'(t)\|.$$

has an inverse function $t(s)$ with $t'(s) = \dfrac{1}{\|\vec{\alpha}'(t(s))\|}$. (Here all the primes are t derivatives.) So

$$T = \frac{d}{ds}\vec{\alpha}(t(s)) = \vec{\alpha}'(t)\cdot t'(s) = \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}.$$

From the Frenet equations, $\frac{d}{ds}T = \kappa N$, so

$$\kappa N = \frac{d^2}{ds^2}\vec{\alpha}(t(s)) = \frac{d}{ds}\frac{\vec{\alpha}'(t(s))}{\|\vec{\alpha}'(t(s))\|}$$

$$= \frac{\vec{\alpha}''}{\|\vec{\alpha}'\|^2} + \left(\frac{d}{ds}\frac{1}{\|\vec{\alpha}'(t(s))\|}\right)\vec{\alpha}'$$

So we know N is in the $\vec{\alpha}', \vec{\alpha}''$ plane with a positive $\vec{\alpha}''$ component.

This means that

colinear

$$
\begin{aligned}
B &= T \times N \\
&= \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|} \times \frac{1}{\kappa}\left(\frac{\vec{\alpha}''}{\|\vec{\alpha}'\|^2} + \underbrace{\left(\frac{d}{ds} \frac{1}{\|\vec{\alpha}'(t(s))\|} \right)}_{\text{a scalar}} \vec{\alpha}' \right) \\
&= \frac{1}{\kappa} \frac{\vec{\alpha}' \times \vec{\alpha}''}{\|\vec{\alpha}'\|^3}.
\end{aligned}
$$

But we know that $\|B\| = 1$ and $\kappa > 0$, so we have

$$
B = \frac{\vec{\alpha}' \times \vec{\alpha}''}{\|\vec{\alpha}' \times \vec{\alpha}''\|} \quad \text{and} \quad \kappa = \frac{\|\vec{\alpha}' \times \vec{\alpha}''\|}{\|\vec{\alpha}'\|^3}.
$$

The torsion formula is homework, but is computed using the Frenet equation

$$
\begin{aligned}
-\tau &= \left\langle \frac{d}{ds} B(t(s)), N \right\rangle \\
&= \left\langle \frac{B'}{\|\vec{\alpha}'\|}, N \right\rangle.
\end{aligned}
$$

$\square$

Example. Consider the helix

$$\vec{\alpha}(t) = (a\cos t,\ a\sin t,\ bt),\quad a > 0$$

We compute

$$\vec{\alpha}' = (-a\sin t,\ a\cos t,\ b)$$

$$\vec{\alpha}'' = (-a\cos t,\ -a\sin t,\ 0)$$

$$\vec{\alpha}''' = (a\sin t,\ -a\cos t,\ 0)$$

$$\|\vec{\alpha}'\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$\vec{\alpha}' \times \vec{\alpha}'' = (-ba\sin t,\ -ba\cos t,\ a^2\sin^2 t + a^2\cos^2 t)$$

$$= (ba\sin t,\ -ba\cos t,\ a^2)$$

$$\|\vec{\alpha}' \times \vec{\alpha}''\| = \sqrt{b^2a^2\sin^2 t + b^2a^2\cos^2 t + a^4}$$

$$= \sqrt{b^2a^2 + a^4} = a\sqrt{a^2+b^2}$$

$$\langle \vec{\alpha}', \vec{\alpha}'' \times \vec{\alpha}''' \rangle = \langle \vec{\alpha}''', \vec{\alpha}' \times \vec{\alpha}'' \rangle$$

$$= ba^2\sin^2 t + ba^2\cos^2 t = ba^2.$$

and

$$(\alpha' \times \vec{\alpha}'') \times \vec{\alpha}' = (ba \sin t, -ba \cos t, a^2) \times (-a \sin t, a \cos t, b)$$

$$= (-b^2 a \cos t - a^3 \cos t, -b^2 a \sin t - a^3 \sin t, \cancel{ba^2 \sin t \cos t} - \cancel{ba^2 \cos t \sin t})$$

$$= (a^2 + b^2)(-a \cos t, -a \sin t, 0)$$

It follows that

$$T = \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|} = \left(-\frac{a}{\sqrt{a^2+b^2}} \sin t, \frac{a}{\sqrt{a^2+b^2}} \cos t, \frac{b}{\sqrt{a^2+b^2}} t \right)$$

$$N = \frac{(\vec{\alpha}' \times \vec{\alpha}'') \times \vec{\alpha}'}{\|\vec{\alpha}'\| \, \|\vec{\alpha}' \times \vec{\alpha}''\|} = (-\cos t, -\sin t, 0)$$

$$B = \frac{\vec{\alpha}' \times \vec{\alpha}''}{\|\vec{\alpha}' \times \vec{\alpha}''\|} = \frac{(b\cancel{a} \sin t, -b\cancel{a} \cos t, a^{\cancel{2}})}{\cancel{a}\sqrt{a^2+b^2}}$$

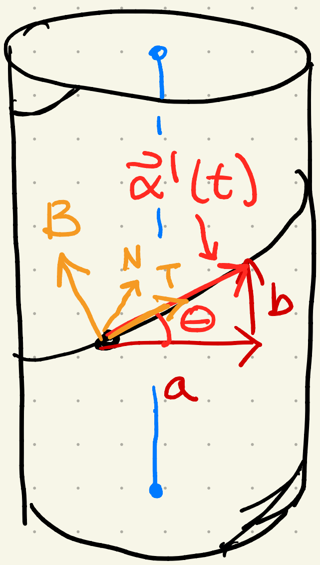
$$= \left(\frac{b}{\sqrt{a^2+b^2}} \sin t, \frac{-b}{\sqrt{a^2+b^2}} \cos t, \frac{a}{\sqrt{a^2+b^2}} \right)$$

and

$$\kappa = \frac{\|\vec{\alpha}' \times \vec{\alpha}''\|}{\|\vec{\alpha}'\|^3} = \frac{a\cancel{\sqrt{a^2+b^2}}}{\left(\sqrt{a^2+b^2}\right)^{\cancel{3}\,2}} = \frac{a}{a^2+b^2}$$

$$\tau = \frac{\langle \vec{\alpha}', \vec{\alpha}'' \times \vec{\alpha}''' \rangle}{\|\vec{\alpha}' \times \vec{\alpha}''\|^2} = \frac{ba^2}{\left(a\sqrt{a^2+b^2}\right)^2} = \frac{b}{a^2+b^2}$$

We see that if θ is the <u>pitch angle</u>



$$\kappa = \frac{\cos\theta}{\|\vec{\alpha}'\|}$$

$$\tau = \frac{\sin\theta}{\|\vec{\alpha}'\|}$$

and $\lim\limits_{b\to 0} \kappa = \dfrac{1}{a}$, $\lim\limits_{b\to\infty} \kappa = 0.$