Math 5200: Active Learning. Exploring a point group.

In class, we saw that if M and N are orthogonal $n \times n$ matrices, then MN is an orthogonal matrix and M^{-1} is an orthogonal matrix. This proves that the collection of orthogonal $n \times n$ matrices forms a "group".[a] This group is called $\mathrm{O}(n)$.

Definition. *A set of matrices $\mathscr{G} \subset \mathrm{O}(n)$ forms a* subgroup *of $\mathrm{O}(n)$ if two properties hold. First, for any $M, N \in \mathscr{G}$, we have $MN \in \mathscr{G}$. Second, if $M \in \mathscr{G}$, then $M^{-1} \in \mathscr{G}$.*

It's a fascinating fact about our 3-dimensional world that there are only a short list of finite subgroups of $\mathrm{O}(3)$. They are generally called the "point groups". This fact controls aspects of chemistry, crystallography, and mathematics. We are now going to explore one of the point groups. Consider the two matrices from the video:

$$\underbrace{A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{permute } \mathbf{e}_1 \to \mathbf{e}_2 \to \mathbf{e}_3} \quad \text{and} \quad \underbrace{B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotate by } \pi \text{ around } \mathbf{e}_3}.$$

1. We are now going to learn about A and B.

   (1) Load the `desmos` matrix calculator (www.desmos.com/matrix) and use it to check that A, $B \in \mathrm{O}(3)$ by computing AA^T and BB^T and verifying that $AA^T = I$ and $BB^T = I$. In the box below, write a description of how you used the calculator.

Solution:

[a] Here, *group* is used in the algebraic sense of MATH 4000/4010, not just colloquially to mean a "collection" or "set" of matrices.

(2) Use the `desmos` matrix calculator to check that

$$B^2 = A^3 = (AB)^3 = I.$$

In the box below, write a brief description of how you used the calculator.

Solution:

2. We are now going to use the calculator to prove there are only 12 unique products of A's and B's, including I. These 12 matrices form a point group we'll call $\mathscr{G}$. We'll do this in a sort of structured way.

   The first three matrices in the group are (of course), I, A, and B themselves:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   (1) We already have three matrices (I, A and B) in our group. There are four matrices which are two-term products of A and B. In dictionary order, they are AA, AB, BA, and BB. However, one of these two-term products is equal to matrix we already have[b] because

$$BB = I.$$

   Compute the three remaining matrix products (in dictionary order) below using `desmos`. Fill in the numerical entries in the blank matrices and also indicate the name of each matrix you're multiplying.

Matrix product name

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \underline{\text{AA}}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad \underline{\text{AB}}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \underline{\text{BA}}$$

[b] That is, one of A, B or I.

(2) We now have six matrices in our group (three one-term products and three two-term products), and we are ready to look for three term products. Multiplying our three two-term products on the right by A and by B (on the right) gives us six possible three-term products. However, not all of these are new: two of the three-term products are equal to one or two-term products we've seen already.

List all six of the possible three-term products in the box below in dictionary order. Then compute each matrix product using `desmos`. When you get a matrix you've seen before, identify the three-term product with a one or two-term product by name. For example, AAA is one of the three term products. But when you compute it, you'll see that

$$AAA = I.$$

So you'll write $AAA = I$ in the box. Record the four new products in the spaces for matrix multiplication below the box and on the next page (in dictionary order)[c].

Solution: We start with AA, AB, and BA, so the six products are AAA, AAB, ABA, ABB, BAA and BAB. Of these, $AAA = I$ and $ABB = A$, so we have seen them before. The other four are new.

Matrix product name

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad \underline{\text{AAB}}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \qquad \underline{\text{ABA}}$$

[c] Note that the left matrix should be one of the two-term products from the last part and the right matrix should be A or B in each product.

Matrix product name

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \underline{\text{BAA}}$$

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad \underline{\text{BAB}}$$

(3) We now have ten matrices in our group[d], including four three-term products. The four three-term products can be multiplied on the right by A or B to give eight four-term products.

List the eight possible four-term products in dictionary order in the box below, and compute each matrix product using `desmos`. Identify six of the products with one, two, or three-term products you've seen before. Write the remaining two (new) matrices in the spaces for matrix multiplication on the next page, in dictionary order.

Solution: The eight four-term products are *AABA*, *AABB*, *ABAA*, *ABAB*, *BAAA*, *BAAB*, *BABA*, and *BABB*. Of these,

$$AABB = AA, \quad ABAB = BAA, \quad BAAA = B$$
$$BAAB = ABA, \quad BABA = AAB, \quad BABB = BA$$

[d] Three one-term products, three two-term products and four three-term products.

Matrix product name

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \underline{\text{AABA}}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \underline{\text{ABAA}}$$

(4) Our group now contains twelve matrix products, including two four-term products. Write down the four possible five-term products in dictionary order in the box below. Carry out the four different matrix multiplications using `desmos`, and identify each of the four matrices as one, two, three, or four-term products that you've already computed.

Solution: The possible five term products are

$$AABAA = BAB, \quad AABAB = ABAA, \quad ABAAA = AB, \quad AABAB = AABA$$