Math 5200: Active Learning.
The General Orbit

Let's start by recalling two definitions:

Definition. *The* orbit *of a vector* $\mathbf{v} \in R^n$ *under the* action *of a group of matrices* $\mathscr{G}$ *is the set*

$$\operatorname{orbit}(\mathbf{v}, \mathscr{G}) = \{\mathbf{x} \text{ s.t. } \mathbf{x} = A\mathbf{v}, A \in \mathscr{G}\}.$$

Definition. *Let*

$$\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The set $\mathscr{V} := \{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ *forms a tetrahedron.*

In the last class, you worked out the effect of each matrix in the "delightful dozen" $\mathscr{G}$ on the vectors in $\mathscr{V}$ and proved that

Proposition. *For each matrix* $M \in \mathscr{G}$ *and each vector* $\mathbf{v} \in \mathscr{V}$*, the vector* $M\mathbf{v} \in \mathscr{V}$*. Thus*

$$\operatorname{orbit}(\mathbf{v}, \mathscr{G}) = \mathscr{V}$$

for any $\mathbf{v} \in \mathscr{V}$*. Further, for any* $M \in \mathscr{G}$*,* $\{M\mathbf{c}, M\mathbf{d}, M\mathbf{e}, M\mathbf{f}\} = \{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$*, which is to that that each* $M \in \mathscr{G}$ *permutes the vectors in* $\mathscr{V}$*.*

We now want to understand $\operatorname{orbit}(\mathbf{x}, \mathscr{G})$ for a general $\mathbf{x}$. To do so, we'll need to write $\mathbf{x}$ in a very special set of coordinates. First, observe that we could scale $\mathbf{x}$ to lie on the surface of the tetrahedron formed by $\mathscr{V} = \{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$.

Definition. *We say* $\mathbf{x} \in \mathbb{R}^n$ *is a* linear combination *of* $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ *if there are* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ *so that* $\mathbf{x} = \sum \lambda_i \mathbf{v}_i$*. If* $0 \leq \lambda_i \leq 1$ *for all* λ_i *and* $\sum \lambda_i = 1$*, we say* $\mathbf{x}$ *is a* convex combination *of the* $\mathbf{v}_i$*, and we call the* λ_i *the* barycentric coordinates *of* $\mathbf{x}$*. We call the set of convex combinations of the* $\mathbf{v}_i$ *the* convex hull *of the* $\mathbf{v}_i$ *and denote it* $\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_{n+1})$.

We can now prove something important.

Proposition. *If* $\mathbf{x}$ *is on the surface of the tetrahedron formed by* $\mathscr{V}$*, then every point in* $\operatorname{orbit}(\mathbf{x}, \mathscr{G})$ *is on the surface of the tetrahedron formed by* $\mathscr{V}$*.*

Proof. Since our tetrahedron is the boundary of the convex hull of $\mathscr{V}$, any point $\mathbf{x}$ on the tetrahedron can be written

$$\mathbf{x} = \lambda_c \mathbf{c} + \lambda_d \mathbf{d} + \lambda_e \mathbf{e} + \lambda_f \mathbf{f}$$

where

$$0 \leq \lambda_c, \ldots, \lambda_f \leq 1 \quad \text{and} \quad \lambda_c + \cdots + \lambda_f = 1.$$

We note that any point written as a convex combination of the points in $\mathscr{V}$ is one the boundary of conv$(\mathscr{V})$ if and only if one (or more) of the λ's are zero. Since we've assumed that $\mathbf{x}$ is on this boundary, we know that at least one of $\lambda_c, \dots, \lambda_f = 0$,

You have worked out the effect of each $M \in \mathscr{G}$ on the set $\mathbf{V}$. Notice that each M is a permutation[a] of the points in $\mathscr{V}$. This means that M^{-1} is also a permutation of $\mathscr{V}$. Thus

$$\begin{aligned} M\mathbf{x} &= \lambda_c M\mathbf{c} + \lambda_d M\mathbf{d} + \lambda_e M\mathbf{e} + \lambda_f M\mathbf{f} \\ &= \lambda_{M^{-1}(c)}\mathbf{c} + \dots + \lambda_{M^{-1}(f)}\mathbf{f}. \end{aligned}$$

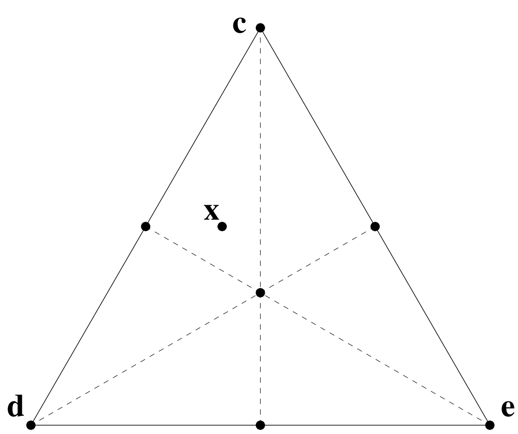
In particular, this means that $M\mathbf{x}$ is a linear combination of the vectors in $\mathscr{V}$ with coefficients that are a permutation of the barycentric coordinates of $\mathbf{x}$. Since these permuted coordinates are still between 0 and 1, still sum to 1, and still have at least one zero coordinate, they are barycentric coordinates for $M\mathbf{x}$, which lies on the boundary of conv$(\mathscr{V})$. $\square$

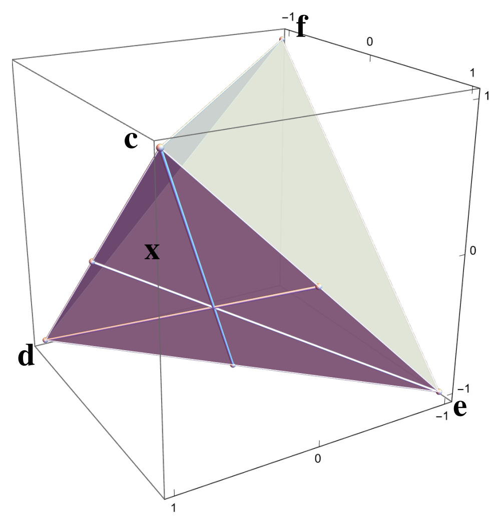
Now suppose that we have a starting point

$$\mathbf{x} = \lambda_c\mathbf{c} + \lambda_d\mathbf{d} + \lambda_e\mathbf{e} + \lambda_f\mathbf{f}$$

whose barycentric coordinates $\lambda = (\lambda_c, \lambda_d, \lambda_e, \lambda_f)$ are in the form $(\lambda_c, \lambda_d, \lambda_e, 0)$. This point must be on the face **cde** of the tetrahedron.

For instance, we'll now show the point $\mathbf{x}$ given in barycentric coordinates by $(1/2, 1/3, 1/6, 0)$ on both the face triangle (left) and the tetrahedron in space (right).





The lines are there to help you get oriented in barycentric coordinates: the lines from vertices to the midpoint of the opposite sides are the lines of points with barycentric coordinates in the form $(x, \frac{1-x}{2}, \frac{1-x}{2})$, $(\frac{1-x}{2}, x, \frac{1-x}{2})$ or $(\frac{1-x}{2}, \frac{1-x}{2}, x)$, where $x \in [0,1]$. As you learned in class before, these meet at the central point with barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

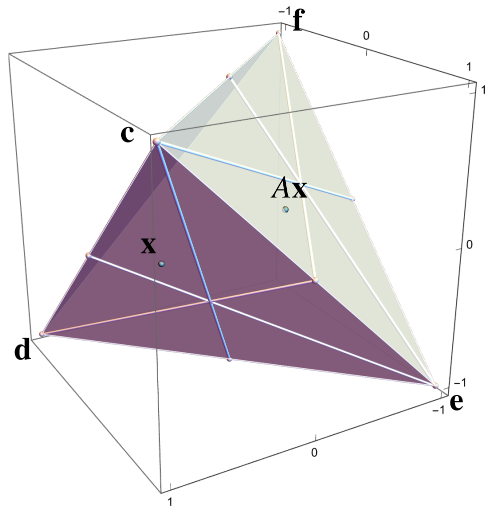
[a] That is, it is a 1-1 and onto map from $\mathscr{V}$ to $\mathscr{V}$.

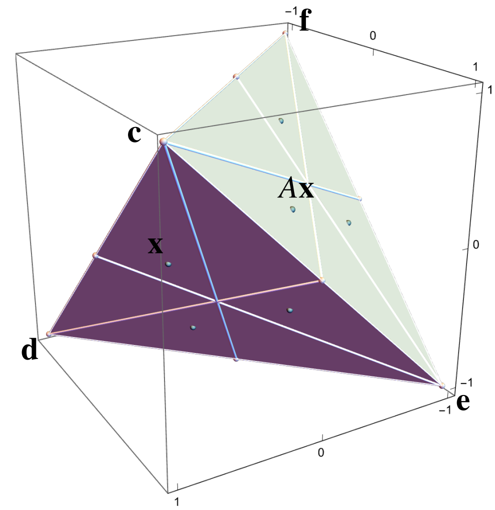
We can now work out

$$A\mathbf{x} = A\left(\frac{1}{2}\mathbf{c}+\frac{1}{3}\mathbf{d}+\frac{1}{6}\mathbf{e}+0\mathbf{f}\right) = \frac{1}{2}A\mathbf{c}+\frac{1}{3}A\mathbf{d}+\frac{1}{6}A\mathbf{e}+0\mathbf{f} = \frac{1}{2}\mathbf{c}+0\mathbf{d}+\frac{1}{3}\mathbf{e}+\frac{1}{6}\mathbf{f}$$

because $A\mathbf{c}=\mathbf{c}$, $A\mathbf{d}=\mathbf{e}$, $A\mathbf{e}=\mathbf{f}$ and $A\mathbf{f}=\mathbf{d}$.

We can now look at $\mathbf{x}$ and $A\mathbf{x}$ on the tetrahedron (left), and the full orbit of $\mathbf{x}$ (right).



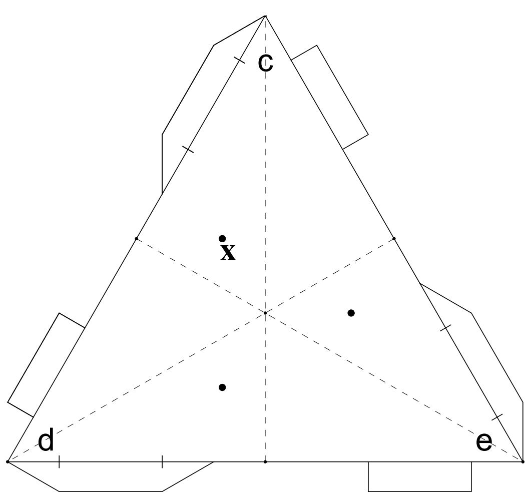


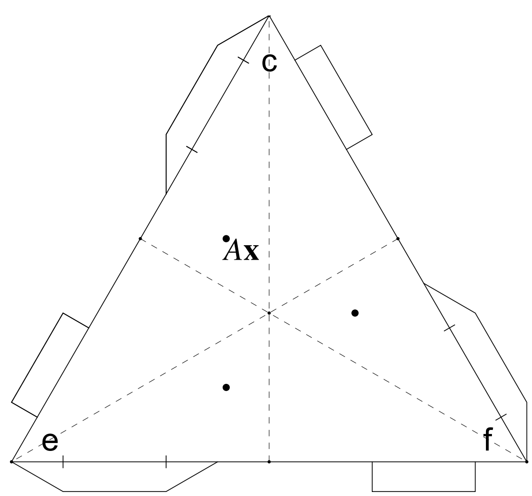
1. We are now going to locate all the points on the orbit on faces of the tetrahedron. All the points with heavy dots are in orbit$(\mathbf{x},\mathscr{G})$. In the diagrams below, label them with the correct matrix. We have already seen that $\triangle\mathbf{cde}$ contains the point

$$\mathbf{x} = \frac{1}{2}\mathbf{c}+\frac{1}{3}\mathbf{d}+\frac{1}{6}\mathbf{e}+0\mathbf{f}$$

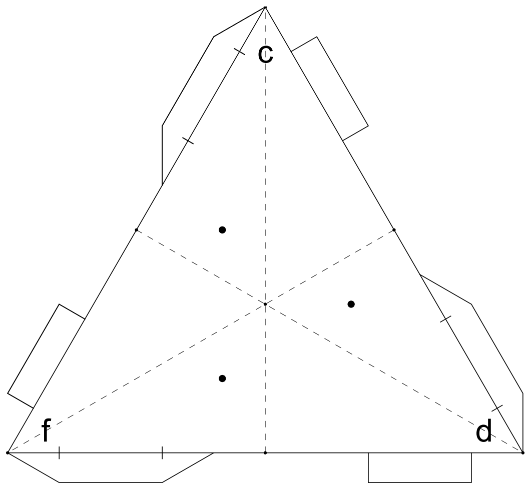
and $\triangle\mathbf{cef}$ which contains the point

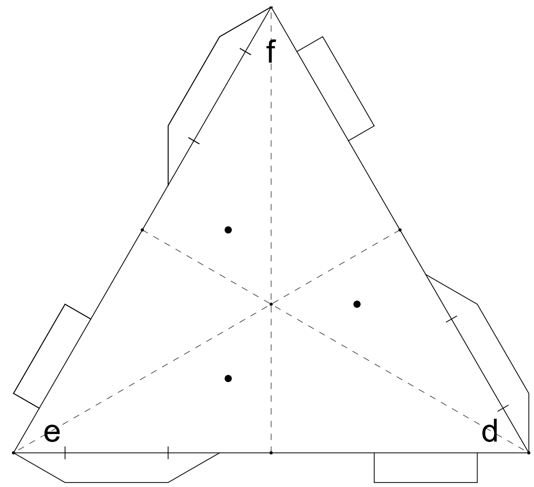
$$A\mathbf{x} = \frac{1}{2}\mathbf{c}+0\mathbf{d}+\frac{1}{3}\mathbf{e}+\frac{1}{6}\mathbf{f}$$



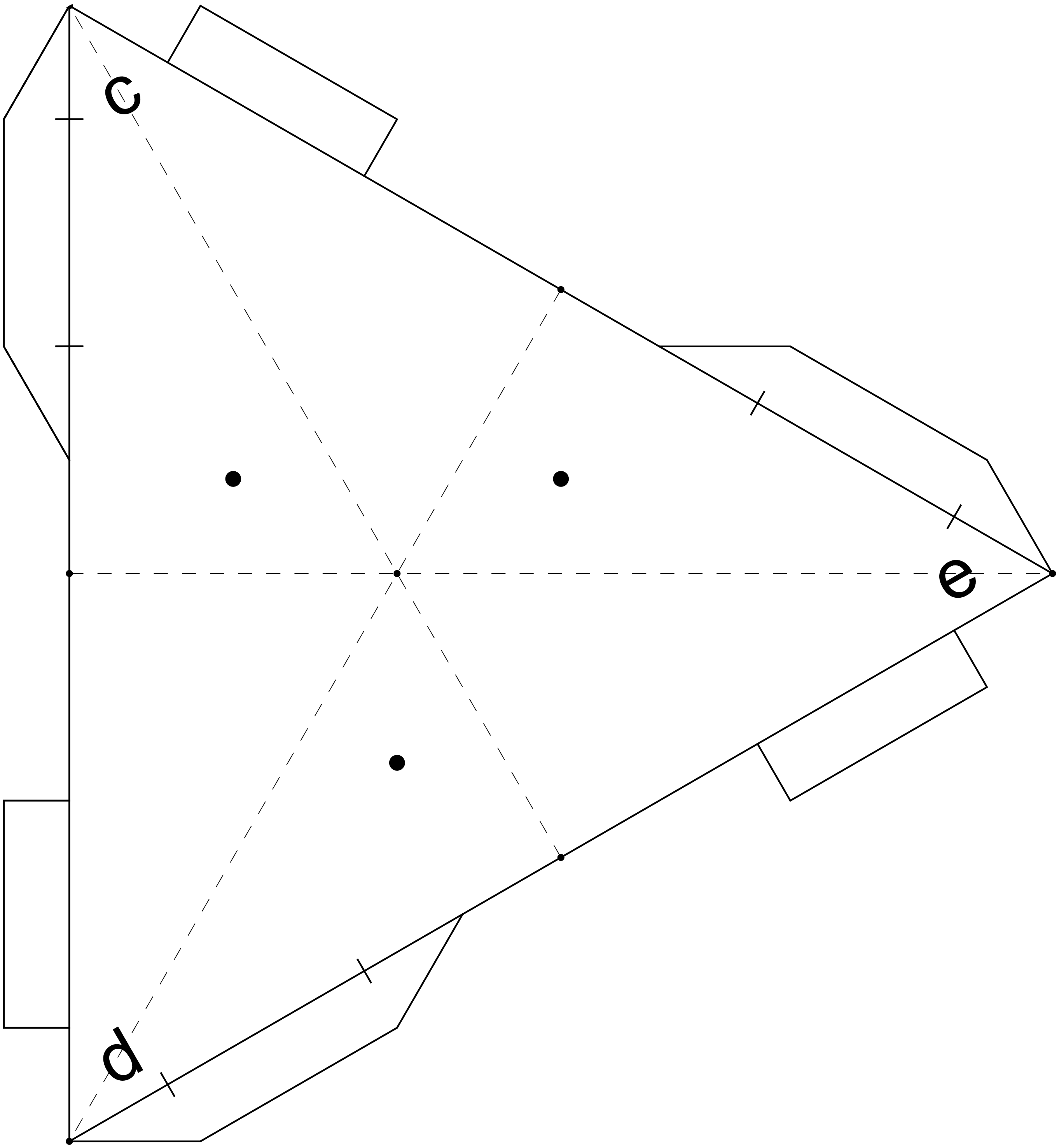


And here are the remaining faces: △**cfd** and △**fed**.





2. The next four pages (printed on heavy stock) contain large versions of the triangles. Cut out each triangle with scissors and use it to assemble the tetrahedron! The slots and tabs are optional, but recommended. When you're done, take some pictures of the result and tape them to this page (in the space below).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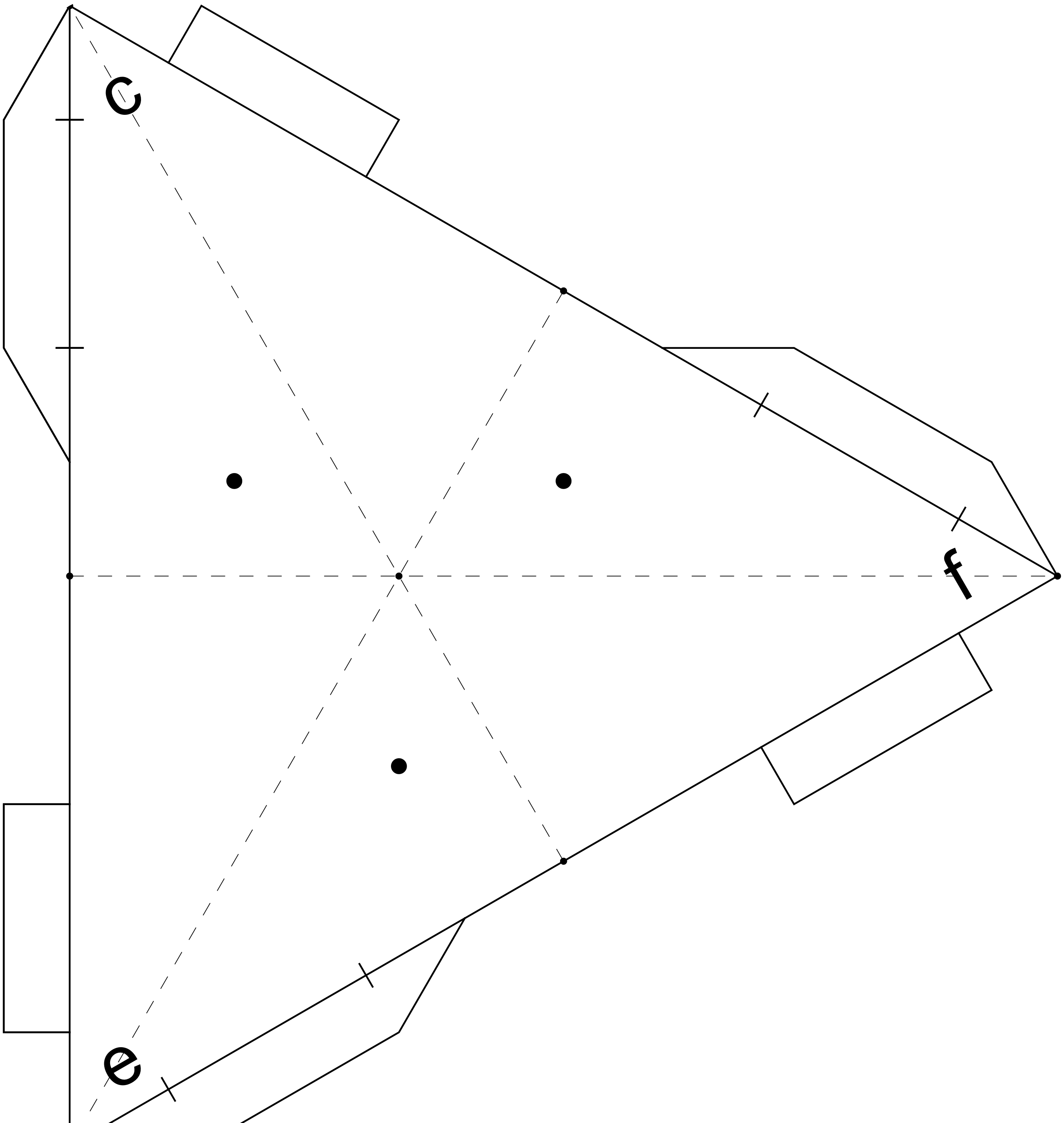
d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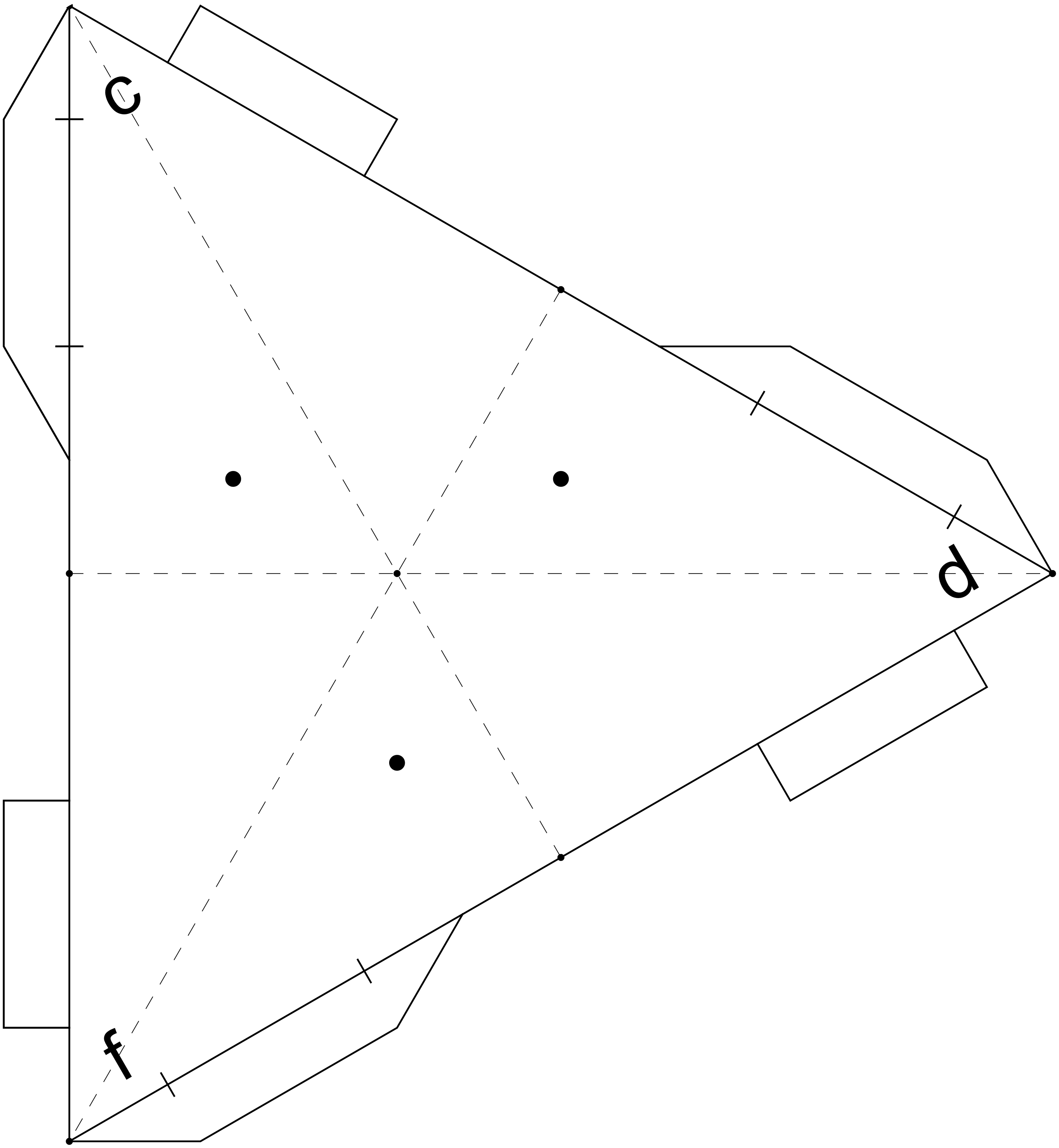
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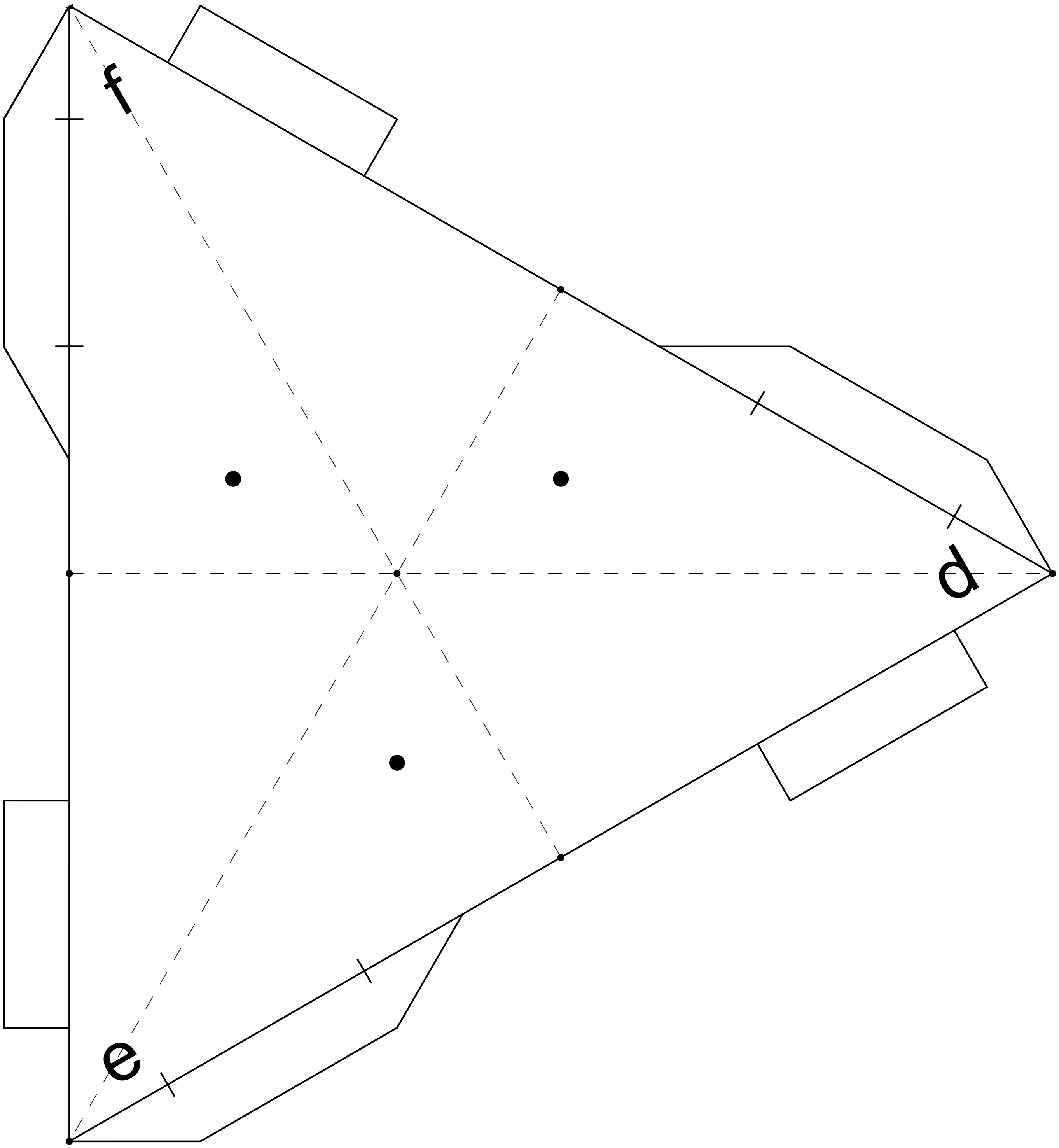
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