# Math 5200: Active Learning.
# Orbits of the delightful dozen.

Recall our definition from the video:

**Definition.** *The* orbit *of a vector* $\mathbf{v} \in R^n$ *under the* action *of a group of matrices* $\mathscr{G}$ *is the set*

$$\text{orbit}(\mathbf{v}, \mathscr{G}) = \{\mathbf{x} \text{ } s.t. \text{ } \mathbf{x} = A\mathbf{v}, A \in \mathscr{G}\}$$

and our definition from class:

**Definition.** *The set* $\mathscr{S} \subset \mathbb{R}^n$ *has a* symmetry *given by an* $n \times n$ *orthogonal matrix* $A$ *if*

$$A\mathscr{S} = \{A\mathbf{x} \text{ } s.t. \text{ } \mathbf{x} \in \mathscr{S}\} = \mathscr{S}$$

Putting these two together, it's immediate that

**Proposition.** *If* $\mathscr{G}$ *is a subgroup of the group of orthogonal matrices* $O(n)$ *and* $\mathbf{v} \in \mathbb{R}^n$*, then every* $M \in \mathscr{G}$ *is a symmetry of* $\text{orbit}(\mathbf{v}, \mathscr{G})$.

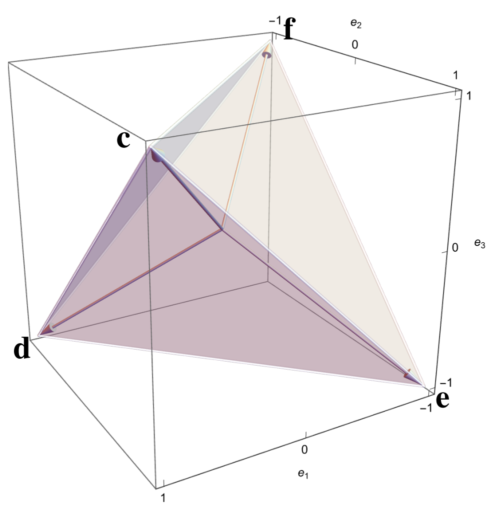
This means that a good way to construct interesting symmetric sets is to combine orbits of points under subgroups of $O(n)$. In fact, if $\mathscr{G}$ is a *finite* subgroup of $O(n)$, we call it a "point group" to emphasize this connection. We note that the number of points in $\text{orbit}(\mathbf{v}, \mathscr{G})$ is at most the number of matrices in $\mathscr{G}$.

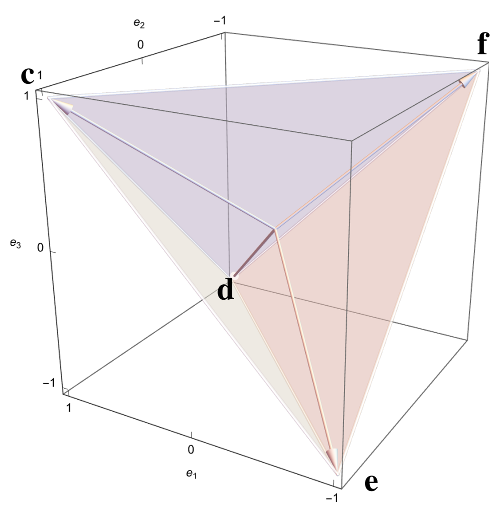
**Definition.** *Let*

$$\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

*The set* $\mathscr{V} := \{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ *forms a tetrahedron.*

Here's a series of pictures showing the tetrahedron:



1. We are now going to use the point group $\mathscr{G}$ from the last homework, so please get out your previous active learning assignment with the list of twelve matrices. Also, load up `desmos` and define the matrices $A$ and $B$ as well as the column vectors $C$, $D$, $E$ and $F$[a].

   We're going to classify each of the matrices in our list of twelve by working out what it does to the vectors $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$. For example,

$$A\mathbf{c} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{c} \qquad A\mathbf{d} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \mathbf{f}$$

$$A\mathbf{e} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \mathbf{d} \qquad A\mathbf{f} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{e}$$

   To stay organized, write the matrices in dictionary order.

| Matrix Name | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $I$ | $\mathbf{c} \mapsto$ | $\mathbf{c}$ | $\mathbf{d} \mapsto$ | $\mathbf{d}$ | $\mathbf{e} \mapsto$ | $\mathbf{e}$ | $\mathbf{f} \mapsto$ | $\mathbf{f}$ |
| $A$ | $\mathbf{c} \mapsto$ | $\mathbf{c}$ | $\mathbf{d} \mapsto$ | $\mathbf{f}$ | $\mathbf{e} \mapsto$ | $\mathbf{d}$ | $\mathbf{f} \mapsto$ | $\mathbf{e}$ |
| $B$ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |
| ______ | $\mathbf{c} \mapsto$ | ______ | $\mathbf{d} \mapsto$ | ______ | $\mathbf{e} \mapsto$ | ______ | $\mathbf{f} \mapsto$ | ______ |

[a] To do matrix-vector multiplication in the `desmos` calculator, write the vector $\mathbf{c}$ as a $3 \times 1$ matrix. Then you can multiply $AC$ and $BC$ (and so forth) by typing these into `desmos` as usual.

2. We know from the video some of the matrices in $\mathscr{G}$ are 120° or 240° rotations around the axes **c**, **d**, **e** and **f**. Using your table above, you can find all of them by figuring out which matrices fix **c**, and which fix **d** and so forth. You should find 9 such matrices total, in four groups of three[b].

   (1) Find all the matrices where $\mathbf{c} \mapsto \mathbf{c}$ and describe what happens to **d**, **e**, and **f**.

   **Example Solution:**

   The matrix $I$ takes $\mathbf{c} \to \mathbf{c}$ and $\mathbf{d} \to \mathbf{d}$, $\mathbf{e} \to \mathbf{e}$, $\mathbf{f} \to \mathbf{f}$.
   The matrix $A$ takes $\mathbf{c} \to \mathbf{c}$ and $\mathbf{d} \to \mathbf{f} \to \mathbf{e} \to \mathbf{d}$.
   The matrix $AA$ takes $\mathbf{c} \to \mathbf{c}$ and $\mathbf{d} \to \mathbf{e} \to \mathbf{f} \to \mathbf{d}$.

   (2) Find all the matrices where $\mathbf{d} \mapsto \mathbf{d}$ and describe what happens to **c**, **e**, and **f**.

   (3) Find all the matrices where $\mathbf{e} \mapsto \mathbf{e}$ and describe what happens to **c**, **d**, and **f**.

   (4) Find all the matrices where $\mathbf{f} \mapsto \mathbf{f}$ and describe what happens to **c**, **d**, and **e**.

[b] The matrix $I$ fixes all vectors (it's a "rotation by 0 degrees around *every* axis") so it's listed in each group. Thus there are only 9 different matrices among the 12 you'll list below.

3. On the other hand, the remaining matrices in $\mathscr{G}$ are 180° rotations around the coordinate axes.[c] These have a different effect on the "axes" **c**, **d**, **e** and **f**. You should find 4 such matrices, in three groups of two[d].

   (1) Find all matrices in $\mathscr{G}$ which send $\mathbf{e}_1 \to \mathbf{e}_1$ and determine what they do to **c**, **d**, **e** and **f**.

   **Example Solution:**

   The matrix $I$ takes $\mathbf{e}_1 \to \mathbf{e}_1$ and $\mathbf{c} \to \mathbf{c}$ and $\mathbf{d} \to \mathbf{d}$, $\mathbf{e} \to \mathbf{e}$, $\mathbf{f} \to \mathbf{f}$.
   The matrix $B$ takes $\mathbf{e}_1 \to \mathbf{e}_1$ and swaps both $\mathbf{c} \leftrightarrow \mathbf{d}$ and $\mathbf{e} \leftrightarrow \mathbf{f}$.

   (2) Find all matrices in $\mathscr{G}$ which send $\mathbf{e}_2 \to \mathbf{e}_2$ and determine what they do to **c**, **d**, **e** and **f**.

   (3) Find all matrices in $\mathscr{G}$ which send $\mathbf{e}_3 \to \mathbf{e}_3$ and determine what they do to **c**, **d**, **e** and **f**.

[c] Of course, the easiest way to do this is to define the vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in `desmos` and multiply them by various combinations of $A$'s and $B$'s from our list of matrices in $\mathscr{G}$ until you win!

[d] Again, the identity matrix is going to be in each group, so there are really only four unique matrices you're going to find among the six total.

4. Notice that we've now shown that $M\mathbf{x} \in \mathscr{V}$ for each $M \in \mathscr{G}$ and $\mathbf{x} \in \mathscr{V}$. In the process, you've proved orbit$(\mathbf{c}, \mathscr{G}) = \mathscr{V}$ and hence that the set $\mathscr{V}$ has every $M \in \mathscr{G}$ as a symmetry.[e] To understand this orbit geometrically, it will help to know another useful fact about orthogonal matrices:

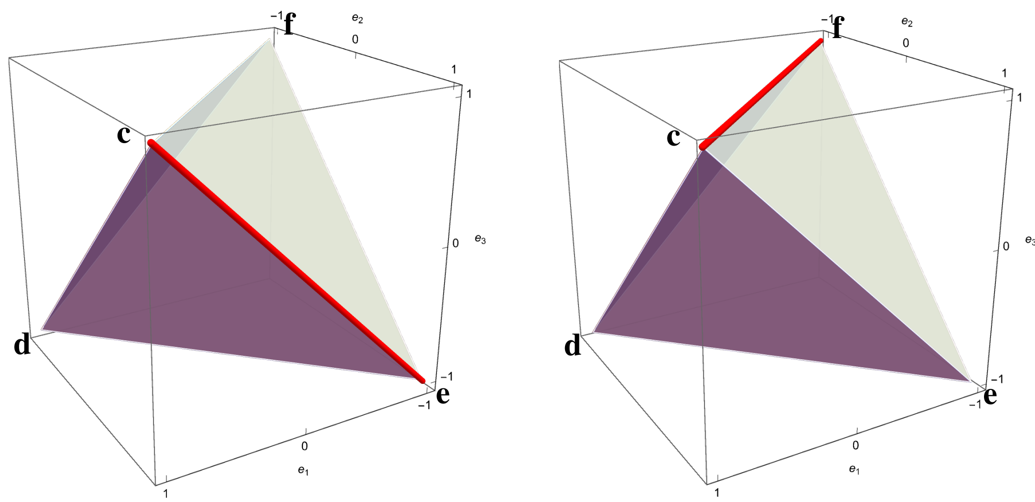
   **Proposition.** *If $A$ is an orthogonal $n \times n$ matrix and $\mathbf{v}$, $\mathbf{w}$ are any vectors in $\mathbb{R}^n$, then $\mathbf{v} \cdot \mathbf{w} = A\mathbf{v} \cdot A\mathbf{w}$. It follows that $\|\mathbf{v}\| = \|A\mathbf{v}\|$ for any $\mathbf{v} \in \mathbb{R}^n$, and hence that*

   $$\|A\mathbf{v} - A\mathbf{w}\| = \|\mathbf{v} - \mathbf{w}\|$$

   *for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and further that $\angle \mathbf{v}\mathbf{w} = \angle A\mathbf{v}A\mathbf{w}$ for any $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$.*

   We now want to prove[f] that for any $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{w} \neq \mathbf{z}$ in $\mathscr{V}$, we have $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{w} - \mathbf{z}\|$.

   Let's consider the edge of the tetrahedron given by $\{\mathbf{c}, \mathbf{e}\}$, shown in red below left and the edge $\{\mathbf{c}, \mathbf{f}\}$ shown in red below right.




   If we can show that there's an orthogonal matrix so that $\{A\mathbf{c}, A\mathbf{e}\} = \{\mathbf{c}, \mathbf{e}\}$, we'll have proved that $\|\mathbf{c} - \mathbf{e}\| = \|\mathbf{c} - \mathbf{f}\|$.

   (1) Find a matrix $M \in \mathscr{G}$ so that $\{M\mathbf{c}, M\mathbf{e}\} = \{\mathbf{c}, \mathbf{f}\}$.

   **Example Solution:**

   Looking at the picture, we can see that rotating around $\mathbf{c}$ by $120°$ or $240°$ ought to do this; looking at our table of matrices which rotate around $\mathbf{c}$, we see that $AA$ takes $\mathbf{c} \to \mathbf{c}$ and $\mathbf{e} \to \mathbf{f}$, so this will do.

[e] We haven't proved that these are the *only* symmetries of $\mathscr{V}$ because, well, it's not true: there are reflection symmetries that aren't in $\mathscr{G}$.

[f] Of course, we could list the $6 = \binom{4}{2}$ pairs of (different) points among the four points in $\mathscr{V} = \{\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$ and just compute each distance; we know the coordinates of $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$. But we want to do this in a way that's more respectful of the symmetry structure.

(2) Find a matrix $M \in \mathscr{G}$ so that $\{M\mathbf{c}, M\mathbf{e}\} = \{\mathbf{c}, \mathbf{d}\}$.

(3) Find a matrix $M \in \mathscr{G}$ so that $\{M\mathbf{c}, M\mathbf{e}\} = \{\mathbf{d}, \mathbf{e}\}$.

(4) Find a matrix $M \in \mathscr{G}$ so that $\{M\mathbf{c}, M\mathbf{e}\} = \{\mathbf{e}, \mathbf{f}\}$.

(5) Find a matrix $M \in \mathscr{G}$ so that $\{M\mathbf{c}, M\mathbf{e}\} = \{\mathbf{f}, \mathbf{d}\}$.