A cell structure for Grassmannians

We want to write $G_n(\mathbb{R}^\infty)$ as a CW-complex. We first recall

Definition. A CW-complex consists of a Hausdorff space K together with a partition of K into $\{e_\alpha\}$ disjoint subsets so that

1) Each e_α is an open cell of dimension $n(\alpha) \geq 0$. Each has a <u>characteristic map</u>

$$f : D^{n(\alpha)} \to K$$

which is a homeomorphism on $\mathrm{int}(D^{n(\alpha)})$ to e_α.

2) Each point in $\overline{e_\alpha}$, but not in e_α is in some e_β of lower dimension.

If the complex has infinitely many e_α,

3) Each point is contained in a finite subcomplex.

4) K is the direct limit of all finite complexes.

We know that every CW complex is paracompact.

We begin by writing $\mathbb{R}^m$ as

$$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^m.$$

Given an n-plane $X \subset \mathbb{R}^m$, we have some integers

$$0 = \dim(X \cap \mathbb{R}^0) \leq \dim(X \cap \mathbb{R}^1) \leq \cdots \leq \dim(X \cap \mathbb{R}^m) = 0.$$

Claim. Two consecutive numbers in this sequence differ by at most 1.

Proof. Consider the sequence

$$0 \to X \cap \mathbb{R}^{k-1} \to X \cap \mathbb{R}^{k} \xrightarrow[\text{k th coord}]{f} \mathbb{R}$$

It is not hard to see that this is exact (we only have to check at $X \cap \mathbb{R}^k$). Thus

$$\dim(X \cap \mathbb{R}^k) = \dim(\ker f) + \operatorname{rank} f$$
$$= \dim(X \cap \mathbb{R}^{k-1}) + \operatorname{rank} f$$

but rank $f = 0$ or 1 since it is a map to $\mathbb{R}$. $\square$

Definition. A <u>Schubert symbol</u> $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a sequence of n integers with

$$1 \leq \sigma_1 < \sigma_2 < \ldots < \sigma_n \leq m.$$

These will locate the possible combinations of n jumps in our sequence of dimensions.

We let

$$e(\sigma) \subset G_n(\mathbb{R}^m) = \text{set of all } X \text{ s.t.}$$

$$\dim(X \cap \mathbb{R}^{\sigma_i}) = i, \ \dim(X \cap \mathbb{R}^{\sigma_i - 1}) = i-1$$

for $i = 1, \ldots, n$.

It is clear that the $e(\sigma)$ cover $G_n(\mathbb{R}^m)$.

Claim. $e(\sigma)$ is an open cell of dimension

$$d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \ldots + (\sigma_n - n).$$

We start with another claim.

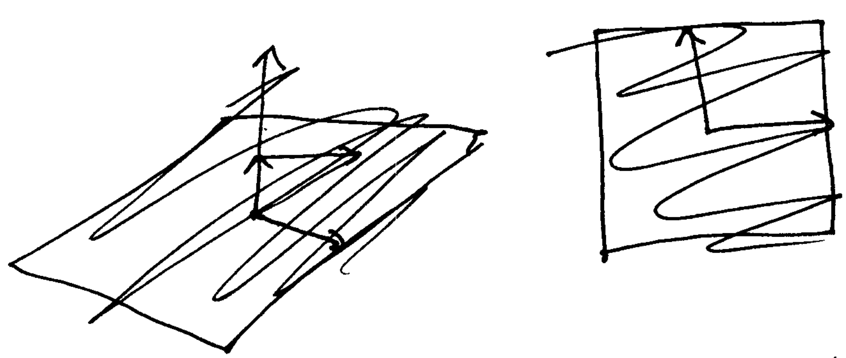
Claim. If $H^k \subset \mathbb{R}^k$ is the ~~positive orthant~~ half-space $(\varepsilon_1, \ldots, \varepsilon_k > 0, 0 \ldots 0)$ then $X \in e(\sigma) \iff X$ has a basis s.t.

$$x_1 \in H^{\sigma_1}, \ldots, x_n \in H^{\sigma_n}$$

Proof. Suppose X has such a basis. Then $x_i \in H^{\sigma_i}$, so the ith coordinate of x_i is > 0. This means that the map $X \cap \mathbb{R}^{\sigma_i} \xrightarrow{\sigma_i \text{th coord}} \mathbb{R}$ has rank 1, and hence $\dim(X \cap \mathbb{R}^{\sigma_i}) > \dim(X \cap \mathbb{R}^{\sigma_i - 1})$, as desired.

Suppose that $X \in e(\sigma)$. ~~We see that~~ ~~$\dim(X \cap \mathbb{R}^{\sigma_i}) = 1$, dim~~ Assume that such a basis $x_1, \dots, x_{i-1}$ exists and we are trying to extend it to x_i.

We know $X \cap \mathbb{R}^{\sigma_i} \xrightarrow[f_{\sigma_i}]{\sigma_i\text{th coord}} \mathbb{R}$ has rank 1, so take any vector in $f_{\sigma_i}^{-1}(1)$. ~~Such a vector can be written as the direct sum of~~ ~~$e_{\sigma_i} = (0, \dots, 0, 1, 0, \dots, 0)$ and~~



Such a vector has kth coord > 1, as desired. It is not in $\operatorname{span}(x_1, \dots, x_{i-1})$ for the same reason. $\square$

We see that $X \in e(\sigma) \Leftrightarrow X$ is the row space of a matrix of the form

$$\begin{bmatrix} * & * & * & 1 & 0 & 0 & - & - & - & 0 \\ * & * & * & * & * & * & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \\ * & * & \cdots & & & * & * & 1 & 0 \cdots & 0 \end{bmatrix}.$$

In fact, these matrices have a nice intersection with $O(m)$.

Lemma. Every n-plane $X \in e(\sigma)$ has a unique orthonormal basis $(x_1, \ldots, x_n)$ which lies in $H^{\sigma_1} \times \cdots \times H^{\sigma_n}$.

Proof. We proceed by induction.

$x_1 \in X \cap \mathbb{R}^{\sigma_1}$, a dim 1 subspace.

Since x_1 is desired to be unit length, there are two possibilities, but only one has σ_1-th coordinate positive. So x_1 is uniquely determined.

Now

$x_i \in X \cap \mathbb{R}^{\sigma_i}$, a dim i subspace,

but $x_i \in \text{span}(x_1, \dots, x_{i-1})^{\perp}$, a dim 1 subspace. There are again two possibilities with unit length, but only one with σ_i-th coord positive.

Definition. Let $e'(\sigma) = V_n^0(\mathbb{R}^m) \cap (H^{\sigma_1} \times \dots \times H^{\sigma_n})$ be the set of orthonormal n-frames with $x_1 \in H^{\sigma_1}, \dots, x_n \in H^{\sigma_n}$. Let $\bar{e}'(\sigma)$ be the orthonormal frames with x_i in the closure $\bar{H}^{\sigma_i}$.

Lemma. $\bar{e}'(\sigma)$ is a closed cell of dim $d(\sigma) = (\sigma_1 - 1) + \dots + (\sigma_n - n)$ with interior $e'(\sigma)$. Further, the map $q: V_n^0 \to G_n$ takes $e'(\sigma)$ homeomorphically onto $e(\sigma)$.

Proof. By induction on n. For $n=1$,

$$\bar{e}'(\sigma_1) = \{ x_1 = (x_{11}, \dots, x_{1\sigma_1}, 0, \dots, 0) \mid |x_1| = 1,\ x_{1\sigma_1} \geq 0 \}$$

This is the closed ~~open~~ hemisphere of dimension $\sigma_1 - 1$ which is homeo. to the disk $D^{\sigma_1 - 1}$.

Now given $u, v \in \mathbb{R}^m$ with $u \neq -v$, let $T(u,v) : \mathbb{R}^m \to \mathbb{R}^m$ be rotation which carries u to v and leaves everything orthogonal to the uv plane fixed.

$$T(u,v)\,x = x - \frac{(u+v)\cdot x}{1 + (u \cdot v)}(u+v) + 2(u \cdot x)v$$

(we can check this by checking that it preserves all $x \perp (u,v)$ and takes $u \to v$ and $v \to u$).

It follows that

1) $T(u,v)x$ is cts in u, v, and x.

2) if $u, v \in \mathbb{R}^k$ then $T(u,v)x \equiv x \pmod{\mathbb{R}^k}$

Now let $b_i \in H^{\sigma_i}$ be the vector $(0,\dots,0,\overset{\sigma_i\text{-th coord}}{1},0,\dots 0)$.

Clearly $(b_1,\dots,b_n) \in e'(\sigma)$. For any other frame $(x_1,\dots,x_n) \in \bar{e}'(\sigma)$ consider

$$T = T(b_n, x_n) \circ T(b_{n-1}, x_{n-1}) \circ \dots \circ T(b_1, x_1).$$

~~This~~ ~~carries~~ $T(b_1,\dots,b_n) \Rightarrow (x_1,\dots,x_n)$.
Claim.

Observe $T(b_1,x_1)$ leaves $b_2,\dots,b_n$ fixed, since they are all orthogonal to b_1 and x_1. In fact

$$T(b_1,x_1), \dots, T(b_{i-1}, x_{i-1})$$

leave b_i fixed.

Now

$$T(b_i, x_i) \quad \text{carries} \quad b_i \to x_i$$

and since the x_i are orthogonal and $b_j \cdot x_i = 0$ for $j > i$, we have that

$$T(b_{i+1}, x_{i+1}), \ldots, T(b_n, x_n) \text{ leave } x_i \text{ fixed.}$$

Now by inductive hypothesis, we know $\bar{e}'(\sigma_1, .., \sigma_n)$ is a closed cell of dimension $(\sigma_1 - 1) + \ldots + (\sigma_n - n)$. We consider the Schubert symbol $\sigma_1, .., \sigma_n, \sigma_{n+1}$ for some $\sigma_{n+1} > \sigma_n$.

Let

$D =$ unit vectors in $\bar{H}^{\sigma_{n+1}}$ with

$$b_1 \cdot u = \ldots = b_n \cdot u = 0.$$

for b_i determined by $\sigma_1, .., \sigma_n$. This is a closed hemisphere of dimension

$$\sigma_{n+1} - n - 1, \quad \text{and a closed cell.}$$

We now define a map

$$f : \bar{e}'(\sigma_1, \dots, \sigma_n) \times D \to \bar{e}'(\sigma_1, \dots, \sigma_{n+1})$$

given by

$$f((x_1, \dots, x_n), u) = (x_1, \dots, x_n, Tu)$$

where T is the rotation from $(b_1, \dots, b_n)$ to $(x_1, \dots, x_n)$ constructed above.

To show $f((x_1, \dots, x_n), u) \in \bar{e}'(\sigma_1, \dots, \sigma_{n+1})$, we must check that

1) Tu is orthogonal to $x_1, \dots, x_n$ and unit.

$$x_i \cdot Tu = Tb_i \cdot Tu \overset{\text{T is a rotation, hence orthogonal}}{=} b_i \cdot u = 0.$$

$$Tu \cdot Tu = u \cdot u = 1$$

2) $Tu \in \bar{H}^{\sigma_{n+1}}$.

We know that $Tu = u \bmod \mathbb{R}^{\sigma_n}$, so the σ_{n+1} st coordinate of u remains positive.

Now we know the $T(b_i, x_i)$ are continuous in b_i, x_i and u, so f is clearly continuous. Further T is invertible by

$$T^{-1} = T(x_1, b_1) \circ \dots \circ T(x_n, b_n)$$

So f^{-1} is well-defined and cts. Thus

$$\bar{e}'(\sigma_1, \dots, \sigma_{n+1}) \cong \bar{e}'(\sigma_1, \dots, \sigma_n) \times D$$

so

$\bar{e}'(\sigma_1, \dots, \sigma_{n+1})$ is homeo. to a closed cell of dimension $(\sigma_1 - 1) + \dots + (\sigma_{n+1} - (n+1))$.

In fact, we can again see by induction that $e'(\sigma) = \text{interior}(\bar{e}'(\sigma))$.

~~Use are~~

Claim: $q|e'(\sigma) : e'(\sigma) \to e(\sigma)$ is a homeomorphism.

We have seen that (Lemma 6.2) every n-plane (in $e(\sigma)$) has a unique basis in $e'(\sigma)$.

So

$$q : e'(\sigma) \to e(\sigma)$$

is 1-1, and onto. We will prove that q takes closed sets in $e'(\sigma)$ to closed sets in $e(\sigma)$.

Let $A \subset e'(\sigma)$ be (relatively) closed. ~~Then~~ in ~~A~~ $e'(\sigma)$ as a subspace of $V_n^0(\mathbb{R}^m)$. Then if we take the closure of A in $V_n^0(\mathbb{R}^m)$,

$$\bar{A} \cap e'(\sigma) = A.$$

Now $\bar{A} \subset \bar{e}'(\sigma)$ is a closed subset of the compact set $\bar{e}'(\sigma)$, so $q\bar{A}$ is compact. Thus $q(\bar{A})$ is closed in $G_n(\mathbb{R}^m)$.

We will now show that

$$q(\bar{A}) \cap e(\sigma) = q(\bar{A})$$

and hence that $q(\bar{A})$ is relatively closed in $e(\sigma)$, completing the proof.

So suppose $(x_1, .., x_n) \in \bar{A} - A$. Then $(x_1, .., x_n) \in \bar{e}'(\sigma) - e'(\sigma)$. We claim the n-plane $X = q(x_1, .., x_n) \notin e(\sigma)$.

Since $(x_1, .., x_n) \notin e'(\sigma)$, one x_i must lie in the boundary of $\bar{H}^{\sigma_i}$. But this boundary is $\mathbb{R}^{\sigma_i - 1}$, so

$$\dim(X \cap \mathbb{R}^{\sigma_i - 1}) \geq i$$

so $X \notin e(\sigma)$.

Thus q is 1-1, onto, and closed, so q is a homeomorphism.

We now have

Theorem. The $\binom{m}{n}$ sets $e(\sigma)$ form the cells of a CW-complex with underlying space $G_n(\mathbb{R}^m)$. Taking the direct limit as $m \to \infty$ yields a CW-structure for G_n.

Proof. We have already found characteristic maps for each $e(\sigma)$. It remains to show

Claim. Each point in $\partial e(\sigma)$ belongs to a cell $e(\gamma)$ of lower dimension.

Now $\bar{e}'(\sigma)$ is compact, so $q\bar{e}'(\sigma) \subset G_n(\mathbb{R}^m)$ is actually $\bar{e}(\sigma)$. So every $X \in \partial e(\sigma)$ is $q(x_1, \ldots, x_n)$ for some $(x_1, \ldots, x_n)$ in $\partial e'(\sigma)$. For these vectors, $x_i \in \mathbb{R}^{\sigma_i}$, so

$$\dim(X \cap \mathbb{R}^{\sigma_i}) \geq i$$

for each i.

Thus ~~each~~ if $(\tau_1, \dots, \tau_n)$ is the Schubert symbol for $(x_1, \dots, x_n)$ we have

$$\tau_i \leq \sigma_i \quad \text{for all } i$$

(we have made at least i "jumps" by dimension σ_i, since $\dim(X \cap \mathbb{R}^{\sigma_i}) = i$).

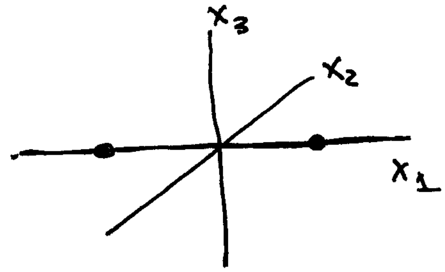
Now $(x_1, \dots, x_n) \notin e'(\sigma)$, so one x_i must lie in the boundary of H^{σ_i}, which is $\mathbb{R}^{\sigma_i - 1}$. Thus one $\tau_i < \sigma_i$, and $d(\tau) < d(\sigma)$, as desired.

What about G_n? Well because of our topology on $\mathbb{R}^\infty$, given any $X \subset G_n$, we can take a basis $x_1, \dots, x_n$ and observe each x_i has only finitely many nonzeros, so all x_i are contained in some (large) $\mathbb{R}^m$!

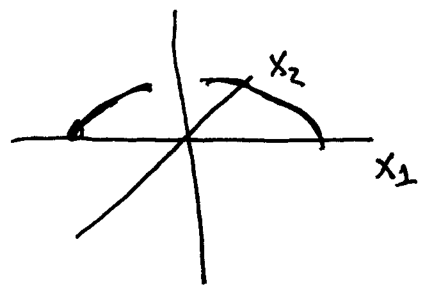
Corollary. $\mathbb{RP}^\infty = G_1(\mathbb{R}^\infty)$ is a CW-complex with one r-cell $e(\{r+1\})$ for each $r>0$.

Proof.

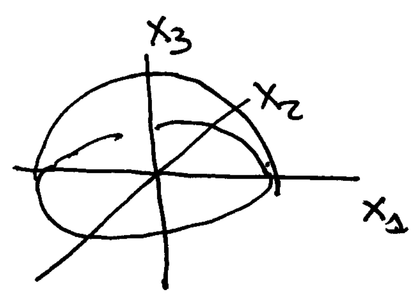
$e(1) = 0$-cell
lines in $\mathbb{R}^1$
$\mathbb{RP}^0$



$e(2) = 1$-cell
lines in $\mathbb{R}^2$
$\mathbb{RP}^1$



$e(3) = 2$-cell
lines in $\mathbb{R}^3$
$\mathbb{RP}^2$



⋮

Now let's count cells in our decomposition.

Definition. ~~The~~ A partition of an integer $r \geq 0$ is an unordered sequence $i_1, \ldots, i_s$ of positive integers with sum r. The number of partitions is $p(r)$.

These grow fast:

r	0	1	2	3	4	5	6	7
$p(r)$	1	1	2	3	5	7	11	15

and in general $p(r) \sim \dfrac{e^{\pi\sqrt{2r/3}}}{4r\sqrt{3}}$ for large r.

To each Schubert symbol with dimension r, there is a corresponding partition

$$\sigma_1 - 1, \ldots, \sigma_n - n \quad \text{of } r$$

denoted $i_1, \ldots, i_s$ (we delete any leading zeros)

Now we must have

$s \leq n$ (since there are n σ_i in the Schubert symbol)

$1 \leq i_1 \leq i_2 \leq \ldots \leq i_s \leq m-n$ (since the σ_i are chosen in $1, \ldots, m$)

So

Corollary. The number of r-cells in $G_n(\mathbb{R}^m)$ is the number of partitions of r into at most n integers, each $\leq m-n$.

So if $n > r$, $m-n > r$ the # of r cells is $p(r)$.