The Cohomology Ring of G_n

We now compute the cohomology ring of G_n with coefficients in $\mathbb{Z}/2$.

Theorem. The cohomology ring $H^*(G_n; \mathbb{Z}/2)$ is a polynomial algebra freely generated by $w_1(\gamma^n), \ldots, w_n(\gamma^n)$.

We start by proving that the ring generated by w_i is free.

Lemma. There are no polynomial relations among the $w_i(\gamma^n)$.

Proof. Suppose $p(w_1(\gamma^n), \ldots, w_n(\gamma^n)) = 0$.
We know that

a) every n-plane bundle has a map into G_n

b) The SW classes of every bundle are the pullbacks of the SW classes of γ^n under this map

So that means that if $p(\omega_1(\gamma^n),..,\omega_n(\gamma^n))=0$ then $p(\omega_1(\xi),..,\omega_n(\xi))=0$ for every n-plane bundle ξ.

So now we build a bundle ξ whose SW classes obey no polynomial relations. Consider γ^1, the line bundle over $G_1=\mathbb{R}P^\infty$. Now we recall that the cohomology of $\mathbb{R}P^n$ was generated ~~by~~ as a polynomial ~~a~~ algebra by the generator a ~~of~~ $H^1(\mathbb{R}P^n)$ with the relation $a^{n+1}=0$.

For $\mathbb{R}P^\infty$, there is no relation.

Further recall

$$w_*(\gamma^1) = 1 + a.$$

We now construct

$$X = \underbrace{\mathbb{RP}^\infty \times \dots \times \mathbb{RP}^\infty}_{n \text{ times}}.$$

By the Kunneth theorem,

$H^1(X; \mathbb{Z}/2)$ is generated by $a_1, \dots, a_n$

and the cohomology ring H^* is the polynomial algebra on these n generators.

Now we take a bundle over X given by the bundle structure of γ_1 over the ith copy of $\mathbb{RP}^\infty$ and call it $\pi_i^* \gamma_1$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & E(\pi_i^* \gamma_1) = E(\gamma_1) \arrow[d, "\pi"] \\
& \underbrace{\mathbb{RP}^\infty \times \dots \times \mathbb{RP}^\infty}_{\text{ith copy}} \times \dots \times \mathbb{RP}^\infty
\end{tikzcd}$$

We can construct

$$\xi = \gamma^1 \times \dots \times \gamma^1 \cong (\pi_1^* \gamma^1) \oplus \dots \oplus (\pi_n^* \gamma^1)$$

which is an n-plane bundle over X.
Now the total SW class of ξ is given by

$$w(\xi) = w(\pi_1^* \gamma^1) \cdots w(\pi_n^* \gamma^1)$$
$$= (1 + a_1) \cdots (1 + a_n).$$

Now if we multiply this out

$$w_1(\xi) = a_1 + a_2 + \dots + a_n$$

$$w_2(\xi) = a_1 a_2 + \dots + a_{n-1} a_n$$
$$= \sum_{\substack{i,j \\ i \neq j}} a_i a_j$$

$$w_n(\xi) = a_1 \cdots a_n.$$

Where have we seen this before?

In algebra class, where we learned that

$$\omega_k(\xi) = k\text{th elementary symmetric function of } a_1, \dots, a_n$$

Fact: The n elementary symmetric polynomials in n variables over a field do not obey any polynomial relations. $\square$

So $\omega_1(\gamma^n), \dots, \omega_n(\gamma^n)$ don't either.

We can now prove the main result. The idea here is that this cohomology ring is the largest one that could be generated by a CW complex with this many cells.

In particular, in cellular homology

rank of vector space generated by cells $\geq$ rank of vector space generated by cycles $\geq$ rank of homology group

We Know that (since $G_n = G_n(\mathbb{R}^\infty)$),

$$\#\text{ of } r\text{-cells of } G_n = \#\text{ of partitions of } r \text{ into at most } n \text{ integers}$$

But we know

$$\operatorname{rank} H_r \geq \#\text{ of monomials of } \cancel{\text{degree}}\ \text{dimension } r \text{ in } \omega_1(\gamma^n), \dots, \omega_n(\gamma^n).$$

since the ~~monomials~~ $\omega_i(\gamma^n)$ obey no polynomial relations. Now such a monomial looks like

$$\omega_1(\gamma^n)^{r_1} \cdots \omega_n(\gamma^n)^{r_n} \quad \text{where } r_1 + 2r_2 + \dots + nr_n = r$$

Claim. # of these monomials = # of at most n elt. partitions

Given such a collection of r_i, write

$$r_1 + 2r_2 + \ldots + nr_n = r$$

$\Rightarrow$

$$\begin{aligned} & r_1 + \\ & + r_2 + r_2 \\ & + r_3 + r_3 + r_3 \qquad\qquad = r \\ & \vdots \\ & r_n + r_n + r_n + \ldots \ldots + r_n \end{aligned}$$

$\Rightarrow$

$$(r_1 + \ldots + r_n) + (r_2 + \ldots + r_n) + \ldots\ldots r_n = r$$

which is a partition of r into at most n elements (delete any zeros). Similarly, given a partition, we can construct a unique monomial by noting that the elts in the partition sum are decreasing from left to right.

This means that all inequalities above were equalities and the SW classes really <u>do</u> generate a free polynomial algebra which is <u>all</u> of the cohomology of the ~~universal bundle~~ Grassmannian G_n. $\square$

Remark. We know that the natural map

$$g : \mathbb{R}P^\infty \times \dots \times \mathbb{R}P^\infty \longrightarrow G_n(\mathbb{R}^\infty)$$

generates a natural homomorphism

$$g^* : H^*(G_n) \to H^*(\mathbb{R}P^\infty \times \dots \times \mathbb{R}P^\infty).$$

By our proof, this maps $H^*(G_n)$ isomorphically onto <u>symmetric</u> polynomials in $a_1, \dots, a_n$.

It is now easy to show:

Uniqueness of SW classes. There is at most one construction $\mathcal{E} \mapsto w(\mathcal{E})$ which assigns to each vector bundle ~~a sequence of $\mathcal{E}$~~ over a paracompact base a sequence of cohomology classes satisfying the SW axioms.

Proof. By our axioms, we know

$$w(\gamma_1^1) = 1 + a, \text{ so } w(\gamma^1) = 1 + a$$

and following the chain,

$$w(\pi_1^*\gamma^1 \oplus \cdots \pi_n^*\gamma^1) = (1+a_1)\cdots(1+a_n)$$

But we know $H^*(G_n) \hookrightarrow H^*(\mathbb{R}P^\infty \times \ldots \times \mathbb{R}P^\infty)$, under the bundle map $\pi_1^*\gamma_1^1 \oplus \ldots \oplus \pi_n^*\gamma_1^1 \to \gamma^n$, so the $w_i(G_n)$ have to generate G_n and they must be as above.

But this means that any natural (axiom 2) SW classes <u>must</u> be our guys!

Problem. 7-A.