Lecture 4. Stiefel-Whitney Classes.

We now present 4 axioms describing desired properties for a system of cohomology classes on a vector bundle. We don't know that any such classes exist yet.

1. To each n-plane ~~vector~~ bundle, ξ there is a sequence of cohomology classes

$$w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2)$$

with $i \in 0, \dots, n$. $w_0 = 1$ and $w_i = 0$ for $i > n$, called the <u>Stiefel-Whitney classes of ξ</u>.

2. (Naturality) If $f : B(\xi) \to B(\eta)$ is the base map of a bundle map $\hat{f} : \xi \to \eta$ then

$$w_i(\xi) = f^* w_i(\eta).$$

3. (Whitney product theorem) If ξ, η have the same base space B, then

we have

$$\omega_k(\xi \oplus \eta) = \sum_{i=0}^{k} \omega_i(\xi) \cup \omega_{k-i}(\eta).$$

4. The Stiefel-Whitney class $\omega_1(\gamma_1^1)$ of the canonical line bundle over the circle $\mathbb{R}P^1$ is nonzero.

We are now going to work out some consequences of these assumptions without worrying about whether the classes exist.

Proposition 1. If $\xi \cong \eta$ then $\omega_i(\xi) = \omega_i(\eta)$.

Proof. Easy consequence of naturality.

Proposition 2. If ε is a trivial bundle then for $i>0$ $\omega_i(\varepsilon) = 0$.

Proof. We know that $E(\varepsilon) = \mathbb{R}^n \times B(\varepsilon)$. Take the map $f: E(\varepsilon) \to \mathbb{R}^n \times \{p\}$ given by

$$f(b, v) = (p, v)$$

this is continuous, and carries each fiber of ~~E~~ ε ~~onto~~ isomorphically onto $\mathbb{R}^n$. Thus this is a bundle map onto the trivial bundle over a point, ε'! Since a point has no homology in dimensions > 0, the S-W classes ~~of~~ $w_i(\varepsilon') = 0$ for $i > 0$. But by naturality, this implies $w_i(\varepsilon) = 0$ for $i > 0$ since

$$w_i(\varepsilon) = f^* w_i(\varepsilon'). \qquad \square$$

By the Whitney product theorem,

Proposition 3. If ε is trivial, $w_i(\varepsilon \oplus \eta) = w_i(\eta)$.

This leads directly to cool consequence:

Proposition 4. If ξ is a Euclidean n-plane bundle with k linearly independent cross sections, then

$$w_n(\xi) = w_{n-1}(\xi) = \cdots = w_{n-k+1}(\xi) = 0.$$

Proof. Take the $\mathbb{R}^K$ sub-bundle of ξ generated by the cross-sections and call it ε. We know that ε is trivial and that

$$\xi = \varepsilon \oplus \underbrace{\varepsilon^{\perp}}_{\text{an } n-K \text{ plane bundle}}$$

But by Prop. 3, this means that

$$w_i(\xi) = w_i(\varepsilon^{\perp}) = \text{the S-W classes of an } n-K \text{ plane bundle}$$

so $w_i(\xi) = 0$ for $i > n-K$.

Consequence. If $w_n(TM) \neq 0$, then M has no nonvanishing vector field. If $w_{n-K}(TM) \neq 0$, then M has at most K nowhere dependent vector fields.

Now suppose that $\xi \oplus \eta$ is trivial. We then have a set of relationships between the SW classes of ξ and η from the Whitney product formulae

For instance, we have

$$w_1(\xi \oplus \eta) = 0 = w_0(\xi) \cup w_1(\eta) + w_1(\xi) \cup w_0(\eta)$$

Now we have to invoke a fact or two about cohomology and cup products. First, we are implicitly assuming that $B(-)$ is connected in all our vector bundles. ~~Next, it is a property of cup products that on a connected~~
Thus $H^0(B(-); \mathbb{Z}/2) = \mathbb{Z}/2$ for all our bundles.
Second, it is a property of cup that if $\alpha \in H^0(M;R) \cong R$, $\beta \in H^k(M;R)$ then

$$\alpha \cup \beta = \overset{\text{multiplication in } R}{\alpha\beta}$$

We can see this easily in DeRham cohomology where cup is wedge and 0-classes are 0-forms (functions).

$$= w_1(\eta) + w_1(\xi). \quad (\text{since } w_0 = 1 \text{ always}).$$

~~So $w_1(\eta) = w_1(\xi)$ (remember we're in $\mathbb{Z}/2$)~~

Similarly (if we eliminate the $\cup$ for cup when it is understood) we have

$$\omega_2(\xi \oplus \eta) = 0 = \omega_2(\xi) + \omega_1(\xi)\,\omega_1(\eta) + \omega_2(\eta).$$

and so forth. Now if we know $\omega_i(\xi)$, observe that we can solve inductively for $\omega_1(\eta), \omega_2(\eta), \ldots, \omega_n(\eta)$ by following this chain of equalities!

We now formalize this process.

Definition. Let $H^{\Pi}(B;\mathbb{Z}/2)$ be a ring whose elements are formal infinite series in the form

$$a = a_0 + a_1 + a_2 + \ldots$$

where $a_i \in H^i(B;\mathbb{Z}/2)$.

$$ab = (a_0b_0) + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \ldots$$

(where the operation is $\cup$ in $H^*(B;\mathbb{Z}/2)$).

and

$$a+b = (a_0+b_0)+(a_1+b_1)+\ \ldots$$

where the addition is in the groups $H^i(B, \mathbb{Z}/2)$.

Lemma. The multiplication ab is commutative and associative.

Proof. We recall that if $\alpha \in H^{p}(B; \mathbb{Z}/2)$, $\beta \in H^{q}(B; \mathbb{Z}/2)$, then

$$\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha.$$

However, we are in $\mathbb{Z}/2$, so the sign is meaningless to us and

$$\alpha \cup \beta = \beta \cup \alpha.$$

Thus our product on H^{Π} inheirits commutativity. Associativity boils down to the observation that the Kth term of abc is given by $\sum_{\substack{x,y,z \\ \text{s.t. } x+y+z=K}} a_x b_y c_z$ in either order.

Definition. The <u>total Stiefel-Whitney class</u> of an n-plane bundle ξ over B is defined to be

$$w(\xi) = 1 + w_1(\xi) + \dots + w_n(\xi) + 0 + \dots$$

in $H^{\Pi}(B; \mathbb{Z}/2)$.

Remark. $w(\xi \oplus \eta) = w(\xi)\,w(\eta)$ by construction.

Lemma. The collection of series that start with 1 forms a commutative group under multiplication.

Proof. Given w, we can construct $\bar{w}$ (the inverse) inductively assuming $w\bar{w} = 1 + 0 + 0 + \dots$.
In fact,

$$\bar{w}_1 = w_1$$

$$\bar{w}_2 + \bar{w}_1 w_1 + w_2 = 0, \text{ so } \bar{w}_2 = w_1^2 + w_2.$$

$$\bar{w}_3 + \bar{w}_2 w_1 + \bar{w}_1 w_2 + w_3 = 0, \text{ so}$$

$$\bar{w}_3 = (w_1^2 + w_2)\,w_1 + w_1 w_2 + w_3 = w_3 + w_1^3.$$

In general,

$$\overline{\omega}_n = \overline{\omega}_{n-1}\,\omega_1 + \overline{\omega}_{n-2}\,\omega_2 + \ldots + \overline{\omega}_1\,\omega_{n-1} + \omega_n,$$

so the induction is clear. $\square$

Remark. We can write this as a power series

$$\begin{aligned}
\overline{\omega} &= \left[1 + (\omega_1+\omega_2+\omega_3+\ldots)\right]^{-1} \\
&= 1 - (\omega_1+\omega_2+\omega_3\ldots) + (\omega_1+\omega_2+\omega_3+\ldots)^2 \\
&\qquad - (\omega_1+\omega_2+\omega_3+\ldots)^3 - \ldots \\
&= 1 - \omega_1 + (\omega_1^2 - \omega_2) + (-\omega_1^3 + 2\omega_1\omega_2 \mp \omega_3) + \ldots
\end{aligned}$$

eventually we can use this to conclude

$$\overline{\omega} = \ldots + \frac{(i_1+\ldots+i_k)!}{i_1!\cdots i_k!}\,\omega_1^{i_1}\cdots\omega_k^{i_k} + \ldots$$

which is pretty cool.

We now know that if ξ and η are bundles over B, then we can solve

$$\omega(\xi \oplus \eta) = \omega(\xi)\,\omega(\eta)$$

for

$$\omega(\eta) = \overline{\omega}(\xi)\,\omega(\xi \oplus \eta).$$

If $\xi \oplus \eta$ is trivial, $\omega(\eta) = \overline{\omega}(\xi)$.

Lemma. (Whitney duality theorem) If $M \subset \mathbb{R}^N$ is a manifold then the Stiefel-Whitney classes of the tangent and normal bundles are related by

$$\omega_i(\nu) = \overline{\omega}_i(TM).$$

We can now actually compute some SW classes!

Example. Since the normal bundle $\nu(S^n)$ is trivial for $S^n \subset \mathbb{R}^{n+1}$, it follows that $w(\nu) = 1$. Thus $\bar{w}(TS^n) = 1$, so $w(TS^n) = 1$ as well.

Conclusion. SW classes are not enough to detect the topology of TS^n.

We now will consider bundles over $\mathbb{R}P^n$. We first recall

$$H^i(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2 \quad \text{for } 0 \leq i \leq n$$
$$= 0 \quad \text{for } i > n$$

Further if a is the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ then the generator of $H^k(\mathbb{R}P^n; \mathbb{Z}/2)$ is a^k.

Example. The canonical line bundle γ_n^1 over $\mathbb{RP}^n$ has total SW class $1+a$.

Proof. Include $j: \mathbb{RP}^1 \to \mathbb{RP}^n$. There is a corresponding bundle map (also called j) $j: \gamma_1^1 \to \gamma_n^1$, so

$$j^* \omega_1(\gamma_n^1) = \omega_1(\gamma_1^1) \neq 0 \text{ (by axiom)}$$

Thus $\omega_1(\gamma_n^1) \neq 0$. But the only elements in $H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ are 0 and a, so $\omega_1(\gamma_n^1) = a$. Since γ_n^1 is a 1-plane bundle, all higher ω_i are automatically zero.

Example. The line bundle γ_n^1 over $\mathbb{RP}^n$ is a subbundle of the trivial $\mathbb{R}^{n+1}$ bundle over $\mathbb{RP}^n$. Let $\gamma^\perp$ be the complement of γ_n^1. Then

$$\omega(\gamma^\perp) = 1 + a + \dots + a^n.$$

Proof. $\gamma_n^1 \oplus \gamma^\perp$ is trivial, so

$$w(\gamma^\perp) = \bar{w}(\gamma_n^1) = (1+a)^{-1} = 1 + a + a^2 + \dots + a^n.$$

Now this bundle is related to $T\mathbb{RP}^n$, but it is not the same (if confused, recall that the "standard" embedding" of $\mathbb{RP}^n$ as a quotient of S^n is <u>not</u> an embedding into $\mathbb{R}^{n+1}$!)

We claim:

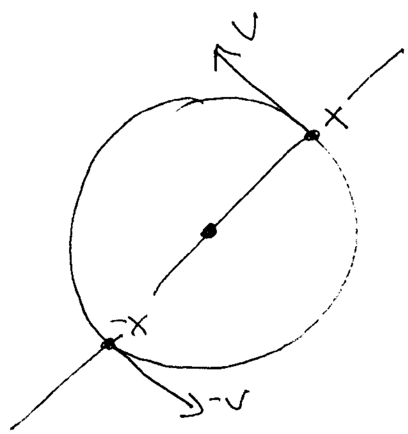
Lemma. $T(\mathbb{RP}^n) \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$.

(What does the rhs mean? Recall that last time we claimed that we could construct a new bundle from any functor on vector spaces. Here's an example. Each fiber of

$$F_b(\mathrm{Hom}(\gamma_n^1, \gamma^\perp)) = \mathrm{Hom}(\underbrace{F_b(\gamma_1^n)}_{\mathbb{R}^1}, \underbrace{F_b(\gamma^\perp)}_{\mathbb{R}^n}).$$

Proof. Let L be a line through the origin (a point in $\mathbb{R}P^n$) in $\mathbb{R}^{n+1}$, intersecting S^n in $\pm x$. Let $L^\perp$ be the complementary n-plane. Let $f: S^n \to \mathbb{R}P^n$ be the canonical double cover map. Consider

$$Df : TS^n \to T(\mathbb{R}P^n).$$



At $\pm x$, Df maps

$$T_x S^n \to T_{\{\pm x\}} \mathbb{R}P^n$$

$$T_{-x} S^n \to T_{\{\pm x\}} \mathbb{R}P^n$$

So we claim $Df(x,v) = Df(-x,-v)$, and TS^n double covers $T(\mathbb{R}P^n)$. Now we can then write

$$T(\mathbb{R}P^n) = \{ (x,v),(-x,v) \in TS^n \times TS^n \}$$

$$= \{ (x,v),(-x,v) \in (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})^2_* \mid x \cdot x = 1,\ x \cdot v = 0 \}.$$

Given such a pair, we can define a linear map $\ell : L \to L^\perp$ by $\ell(x) = v$.

(And given such a linear map, we can define v.)

So $T_{\{\pm x\}} \mathbb{R}P^n$ is canonically isomorphic to $\mathrm{Hom}(L, L^\perp)$. Since the iso. is canonical, this implies it is cts. in $L, L^\perp$, and $T(\mathbb{R}P^n) \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$. $\square$.

It would be great if this let us compute $\omega(T(\mathbb{R}P^n))$. But we can't quite do this yet directly from our description of $T(\mathbb{R}P^n)$. However, we have

Theorem. If $\varepsilon_1 = \mathbb{R} \times \mathbb{R}P^n$, then

$$T(\mathbb{R}P^n) \oplus \varepsilon_1 = \gamma_n^1 \oplus \gamma_n^1 \oplus \cdots \oplus \gamma_n^1.$$

The proof of this theorem is absolutely cute!

Proof. The bundle $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is a 1-plane bundle, since the fiber is $\mathrm{Hom}(\mathbb{R}, \mathbb{R})$ which is $\mathbb{R}^* = \mathbb{R}^1$. Further it has a nonvanishing cross-section so it is the trivial 1-plane bundle ε^1.

So

$$\begin{aligned} T(\mathbb{R}P^n) \oplus \varepsilon^1 &= \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \\ &= \mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \\ &= \mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}) \\ &= \mathrm{Hom}(\gamma_n^1, \varepsilon^1 \oplus \cdots \oplus \varepsilon^1) \\ &= \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \oplus \cdots \oplus \mathrm{Hom}(\gamma_n^1, \varepsilon^1). \end{aligned}$$

But γ_n^1 is Euclidean (since it is a 1-plane bundle), so fiber by fiber, $\mathrm{Hom}(F_b(\gamma_n^1), F_b(\varepsilon^1))$ $= \mathrm{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R} = F_b(\gamma_n^1)$.

canonically

so $\mathrm{Hom}(\gamma_n^1, \varepsilon^1) \cong \gamma_n^1$. $\square$

Corollary.

$$
\begin{aligned}
w(T(\mathbb{R}P^n)) &= w(T(\mathbb{R}P^n) \oplus \epsilon^1) \\
&= w(\gamma_n^1 \oplus \cdots \oplus \gamma_n^1) \\
&= w(\gamma_n^1) \cdots w(\gamma_n^1) \\
&= (1+a) \cdots (1+a) \\
&= (1+a)^{n+1} \\
&= 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \cdots + \binom{n+1}{n} a^n
\end{aligned}
$$

(Remember that $a^{n+1} = 0$ in $H^*(\mathbb{R}P^n)$.)

Examples.

$w(\mathbb{R}P^2) = 1 + a + a^2$

$w(\mathbb{R}P^3) = 1$

$w(\mathbb{R}P^4) = 1 + a + a^4$

$\vdots$

0	1					
1	1	1				
2	1	0	1			
3	1	1	1	1		
4	1	0	0	0	1	
5	1	1	0	0	1	1

Corollary. The only projective spaces that can be parallelizable are $\mathbb{R}P^{2^n-1}$, since these are the only $\mathbb{R}P^k$ with $w(\mathbb{R}P^k) = 1$.

Proof. Recall $(a+b)^2 \equiv a^2 + b^2 \mod 2$. So since we are in $\mathbb{Z}/2\mathbb{Z}$ coefficients, we have

$$(1+a)^{2^r} = 1 + a^{2^r}$$

so if $2^r = n+1$, then

$$w(T(\mathbb{R}P^n)) = (1+a)^{n+1} = (1+a)^{2^r} = 1 + a^{2^r} = 1 + a^{n+1} = 1.$$

Now if $n+1$ is <u>not</u> a power of 2, then $n+1 = 2^r m$ with $m > 1$ odd. We observe

$$\begin{aligned} w(T(\mathbb{R}P^n)) &= (1+a)^{n+1} \\ &= (1 + a^{2^r})^m \\ &= 1 + m a^{2^r} + \binom{m}{2}(a^{2^r})^2 + \dots \end{aligned}$$

But $2^r < n+1$, so $a^{2^r} \neq 0$, and m odd, so $m a^{2^r} \neq 0$ and $w(T(\mathbb{R}P^n)) \neq 1$. $\square$

Remark. We will prove $\mathbb{RP}^1, \mathbb{RP}^3, \mathbb{RP}^7$ are parallelizable in a minute. But it is known that $\mathbb{RP}^{15}, \mathbb{RP}^{31}, \ldots$ are <u>not</u> parallelizable for more subtle reasons.

Theorem. Suppose $\exists$ a bilinear product
$$p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$
without zero divisors. Then $\mathbb{RP}^{n-1}$ is parallelizable.

Proof. Let $b_1, \ldots, b_n$ be the standard basis for $\mathbb{R}^n$. There is a map
$$A : y \mapsto p(y, b_1) \text{ from } \mathbb{R}^n \to \mathbb{R}^n.$$
This map is linear and has no kernel (other than $y = 0$) since p has no zero divisors so it is an isomorphism. Now we define n maps $V_i : \mathbb{R}^n \to \mathbb{R}^n$ by
$$V_i(p(y, b_1)) = p(y, b_i)$$
or
$$V_i(x) = p(A^{-1}(x), b_i).$$

We claim that

$$V_1(x), \ldots, V_n(x) \quad \text{are lin. indep. for } x \neq 0$$

and $V_1(x) = x$.

Suppose

$$a_1 V_1(x) + \ldots + a_n V_n(x) = 0, \quad \text{all } a_i \text{ not zero}$$

Well,

$$a_1 V_1(x) + \ldots + a_n V_n(x) = a_1 p(A^{-1}(x), b_1) + \ldots + a_n p(A^{-1}(x), b_n)$$

$$= p(A^{-1}(x), a_1 b_1 + \ldots + a_n b_n),$$

so we have found zero divisors $A^{-1}(x)$ ($\neq 0$ since A is an isomorphism and $x \neq 0$) and $\sum a_i b_i$ ($\neq 0$ since not all a_i are zero).

Further,

$$V_1(x) = p(A^{-1}(x), b_1) = A(A^{-1}(x)) = x.$$

Claim: $v_2, \dots, v_n$ give rise to $n-1$ nowhere dependent cross sections of

$$T(\mathbb{RP}^{n-1}) \cong \mathrm{Hom}(\gamma^1_{n-1}, \gamma^\perp).$$

Let L be some line through the origin. We define a map

$$\bar{v}_i : L \to L^\perp$$

by for $x \in L$, let $\bar{v}_i(x) = \pi(v_i(x))$ where $\pi : \mathbb{R}^n \to L^\perp$ is orthogonal projection down L.

1) $\bar{v}_1 = 0$. Since $v_1(x) = x$ is on L.

2) $\bar{v}_2, \dots, \bar{v}_n$ are lin. indep.

If not, then

$$a_2 \bar{v}_2 + \dots + a_n \bar{v}_n = 0$$

or

$$a_2(v_2 + b_2 x) + \dots + a_n(v_n + b_n x) = 0$$

$$(a_2 b_2 v_1 + \dots + a_n b_n v_1) + a_2 v_2 + \dots + a_n v_n = 0$$

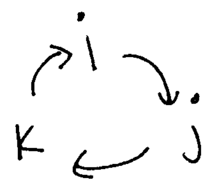
and $v_1, \dots, v_n$ are linearly dependent. $\times\!\!\!\times$

Thus $\bar{v}_2, \dots, \bar{v}_n$ trivialize $\mathrm{Hom}(\gamma^1_{n-1}, \gamma^\perp)$ and hence $T(\mathbb{R}P^{n-1})$ is trivial. $\square$

Examples.

$\mathbb{R}^2$ = complex numbers

$\mathbb{R}^4$ = quaternions



Quaternionic product of pure quaternions

$q = q_1 i + q_2 j + q_3 k$

$p = p_1 i + p_2 j + p_3 k$

can be written in terms of $\mathbb{R}^3$ vectors

$\vec{q} = (q_1, q_2, q_3)$, $\vec{p} = (p_1, p_2, p_3)$

$$qp = \vec{q}\cdot\vec{p} + (\vec{q}\times\vec{p})_1 i + (\vec{q}\times\vec{p})_2 j + (\vec{q}\times\vec{p})_3 k.$$

$\mathbb{R}^7$ = octonions or Cayley numbers.

(nb. Look this up!)

Immersions.

Which $\mathbb{R}P^n$ can be embedded in $\mathbb{R}^{n+k}$?

By Whitney duality, if M^n immerses in $\mathbb{R}^{n+k}$ then

$$w_i(\nu) = \bar{w}_i(M)$$

implies $\bar{w}_i(M) = 0$ for $i > k$.

Example. Consider $\mathbb{R}P^9$.

$$w(\mathbb{R}P^9) = (1+a)^{10}$$
$$= 1 + a^2 + a^8.$$

$\bar{w}(\mathbb{R}P^9)$ is obtained by solving

$$\bar{w}_0 = 1$$
$$\bar{w}_1 = w_1 = 0.$$
$$\bar{w}_2 = \bar{w}_1 w_1 + w_2 = 1.$$
$$\bar{w}_3 = \cancel{\bar{w}_2 w_1} + \cancel{\bar{w}_1 w_2} + \cancel{w_3} = 0$$
$$\bar{w}_4 = \cancel{\bar{w}_3 w_1} + \bar{w}_2 w_2 + \cancel{\bar{w}_1 w_3} + \cancel{w_4} = 1$$

$$\bar{\omega}_5 = \cancel{\bar{\omega}_4 \omega_1} + \cancel{\bar{\omega}_3 \omega_2} + \cancel{\bar{\omega}_2 \omega_3} + \cancel{\bar{\omega}_1 \omega_4} + \cancel{\omega_5}$$
$$= 0$$

$$\bar{\omega}_6 = \cancel{\bar{\omega}_5 \omega_1} + \bar{\omega}_4 \omega_2 + \cancel{\bar{\omega}_3 \omega_3} + \cancel{\bar{\omega}_2 \omega_4} + \cancel{\bar{\omega}_1 \omega_5} + \cancel{\omega_6}$$
$$= 1$$

$$\bar{\omega}_7 = \cancel{\bar{\omega}_6 \omega_1} + \cancel{\bar{\omega}_5 \omega_2} + \cancel{\bar{\omega}_4 \omega_3} + \cancel{\bar{\omega}_3 \omega_4} + \cancel{\bar{\omega}_2 \omega_5}$$
$$+ \cancel{\bar{\omega}_1 \omega_6} + \cancel{\omega_7} = 0.$$

$$\bar{\omega}_8 = \cancel{\bar{\omega}_7 \omega_1} + \overset{1}{\bar{\omega}_6} \omega_2 + \cancel{\bar{\omega}_5 \omega_3} + \cancel{\bar{\omega}_4 \omega_4}$$
$$+ \cancel{\bar{\omega}_3 \omega_5} + \cancel{\bar{\omega}_2 \omega_6} + \cancel{\bar{\omega}_1 \omega_7} + \overset{1}{\omega_8} = 0.$$

So

$$\boxed{\bar{\omega}(\mathbb{RP}^9) = 1 + a^2 + a^4 + a^6.}$$

This means that if $\mathbb{RP}^9$ immerses in $\mathbb{R}^{9+k}$, then $k \geq 6$. Cool!

The strongest results along these lines come from $\mathbb{RP}^{2^n}$.

If $n = 2^r$,

$$
\begin{aligned}
\omega(\mathbb{R}P^n) &= \omega(\mathbb{R}P^{2^r}) \\
&= (1+a)^{2^r+1} \\
&= (1+a)^{2^r}(1+a) \\
&= (1+a^{2^r})(1+a) \\
&= (1+a^{n})(1+a) \\
&= 1+a+a^n
\end{aligned}
$$

Then we can compute

$$
\begin{aligned}
\overline{\omega}_i(\mathbb{R}P^n) &= \sum_{k=i-1}^{0} \overline{\omega}_k \, \omega_{i-k} \\
&= \begin{cases} \overline{\omega}_{i-1} & i \in 1, \ldots, n-1 \\ \overline{\omega}_{i-1} + \omega_n & i = n \end{cases} \\
&= \begin{cases} 1 & i \in 1, \ldots, n-1 \\ 0 & i = n \end{cases}
\end{aligned}
$$

so

$$\overline{\omega}(\mathbb{R}P^n) = 1 + a + a^2 + \ldots + a^{n-1}$$

Theorem. If $\mathbb{RP}^{(2^r)}$ can be immersed in $\mathbb{R}^{2^r+K}$, then $K \geq 2^r - 1$, or

$\mathbb{RP}^{(2^r)}$ cannot be immersed in any $\mathbb{R}^n$ smaller than $\mathbb{R}^{2(2^r)-1}$.

Proof. We just gave it.

Remark. Whitney Immersion Theorem shows that any M^n immerses in $\mathbb{R}^{2n-1}$, so this example shows that this theorem is optimal.

Stiefel-Whitney Numbers.

At the moment, the Stiefel-Whitney classes depend on the cohomology ring of the manifold, so there is no way to directly compare the classes of different manifolds. Let's fix that!

Let M be a closed, smooth n-manifold, and let

$$\mu_M \in H_n(M; \mathbb{Z}/2)$$

be the <u>fundamental class</u> of M, (or <u>top class</u>).

Suppose

$$r_1 + 2r_2 + \cdots + nr_n = n$$

then given any vector bundle ξ over M, we can form

$$\omega_1(\xi)^{r_1} \cdots \omega_n(\xi)^{r_n} \in H^n(M; \mathbb{Z}/2).$$

<u>Definition</u>. $\langle \omega_1(TM)^{r_1} \cdots \omega_n(TM)^{r_n}, \mu_M \rangle$

or $\omega_1^{r_1} \cdots \omega_n^{r_n}[M]$ is the <u>Stiefel-Whitney number</u> associated with $\omega_1^{r_1} \cdots \omega_n^{r_n}$.

Two n-manifolds have the <u>same</u> SW numbers if these agree for all such monomials with $\sum k r_k = n$.

<u>Example.</u> SW numbers of $\mathbb{RP}^n$.

We know $w(\mathbb{RP}^n) = (\cancel{\ }1 + a)^{n+1}$

$$= 1 + (n+1)a + \dots + (n+1)a^n$$

where the intermediate guys are binomial coefficients in $\mathbb{Z}/2$.

If n is even,

$$w_n[\mathbb{RP}^n] = (n+1)a^n \neq 0,$$

$$w_1^n[\mathbb{RP}^n] = (n+1)a^n \neq 0.$$

(If n is a power of 2, $w(\mathbb{RP}^n) = 1 + a + a^n$, and this is it for the SW numbers.)

If n is odd, let $n = 2k-1$, so

$$w(\mathbb{RP}^n) = (1+a)^{2k} = (1+a^2)^k,$$

and $w_j(\mathbb{RP}^n) = 0$ when j is odd.

Now suppose

$$r_1 + 2r_2 + \dots + nr_n = n = \text{odd}.$$

Then one of the terms, say jr_j is odd, so j and r_j are both odd. But $w_j(\mathbb{R}P^n) = 0$, so $w_j^{r_j}(\mathbb{R}P^n) = 0$ and thus the entire monomial

$$w_1^{r_1} \cdots w_n^{r_n}(\mathbb{R}P^n) = 0.$$

So all SW numbers vanish for $\mathbb{R}P^{odd}$.

You would think from this that SW numbers are pretty weak. However, the follow thms show that this is <u>not</u> the case.

Theorem. [Pontrjagin] If M^n is the bounday of a smooth compact $n+1$ manifold B, then the SW numbers of M are all zero.

Proof. Let $\mu_B \in H_{n+1}(B, M)$ be the top class of B. We know

$$\partial : H_{n+1}(B, M) \to H_n(M)$$

takes μ_B to μ_M. Further, for any $v \in H^n(M)$,

$$\langle v, \partial \mu_B \rangle = \langle \delta v, \mu_B \rangle$$

(where $\delta : H^n(M) \to H^{n+1}(B, M)$ is the map induced by ∂ under the Hom functor that takes homology to cohomology.)

Now

TB restricted to M has TM as a sub-bundle.

If we let TB be a Euclidean bundle, there is a unique outward direction on $M = \partial B$, so

$$TB|_M \cong TM \oplus \varepsilon^1$$

so the SW classes of $TB|_M$ are equal to the SW classes of TM.

Now the inclusion $M \hookrightarrow B$ gives us the (cohomology) exact sequence of the pair (B, M)

$$H^n(B) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(B, M)$$

Now we observe that by naturality, since the inclusion $M \hookrightarrow B$ is covered by the bundle map from $TB|_M \to TB$, we know that the SW classes of $TB|_M$ are the restrictions of the SW classes of TB to M. These live in $H^n(B)$.

In particular, if $\omega_1^{r_1} \cdots \omega_n^{r_n} \in H^n(M)$ is a product of SW classes, of TM, it is the image under i^* of corresponding classes $\omega_1^{r_1} \cdots \omega_n^{r_n} \in H^n(B)$.

Thus $\delta(\omega_1^{r_1} \cdots \omega_n^{r_n}) = 0$ for all such monomials.

And for any SW number of M,

$$\langle \omega_1^{r_1} \cdots \omega_n^{r_n}, \mu_M \rangle = \langle \omega_1^{r_1} \cdots \omega_n^{r_n}, \partial \mu_B \rangle$$

$$= \langle \delta(\omega_1^{r_1} \cdots \omega_n^{r_n}), \mu_B \rangle$$

$$= 0. \qquad \square$$

Theorem. [Thom] If all SW numbers of M are zero, then $M = \partial B$ for some smooth compact B.

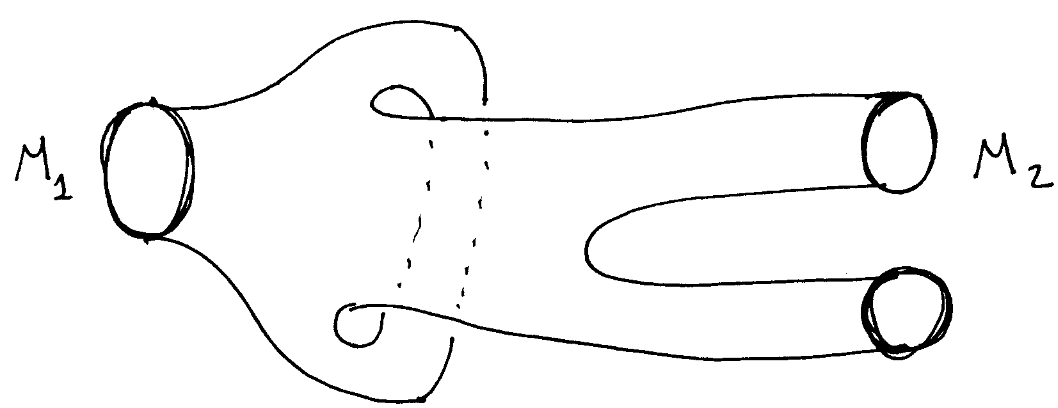
Proof. Beyond scope of this class. (Hard.)

Example. $\mathbb{R}P^{\text{odd}}$ has all SW numbers 0, so $\mathbb{R}P^{\text{odd}}$ is a boundary.

$\mathbb{R}P^1 = S^1 = \partial D^2$

$\mathbb{R}P^3 = \partial\,(\text{what?})$

Definition. Smooth closed manifolds M_1 and M_2 are (unoriented) <u>cobordant</u> if $M_1 \sqcup M_2$ is the boundary of a smooth compact $(n+1)$ manifold.



Corollary. If M_1, M_2 are smooth closed n-manifolds, then M_1 and M_2 are cobordant $\Leftrightarrow$ all SW #s are equal.

Proof. All SW #s of $M_1 \sqcup M_2$ are even, hence zero. Now apply Thom.

Problems. 4A-4C.