

Continuous and Discrete Differential Geometry,
Notes 1: Plane Curvature and Curve Shortening

Definition. Parametrized curve,
regular parametrization.

Definition. T, N, κ .

Theorem. If $T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$ and

$N(t) := T(t)^\perp$, then

$$\kappa(t) := \frac{\langle \gamma'(t)^\perp, \gamma''(t) \rangle}{\|\gamma'(t)\|^3}$$

If $\|\gamma'(t)\| \equiv 1$, simplifies to

$$\kappa(t) = \langle N(s), T'(s) \rangle.$$

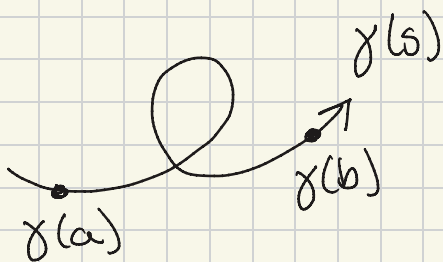
Some neat results about $\chi(t)$.

Proposition. If $\varphi(s)$ = angle between $T(s)$ and horizontal, then

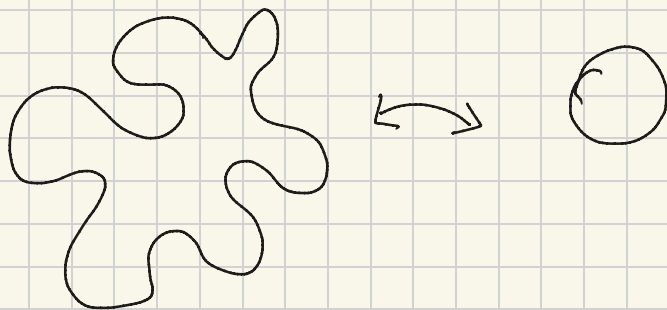
$$\int_a^b \chi(s) ds = \varphi(b) - \varphi(a).$$

Proof. Differentiate $\cos(\varphi(s)) = \langle T(s), \vec{x} \rangle$
and solve to show $\varphi'(s) = \chi(s)$.

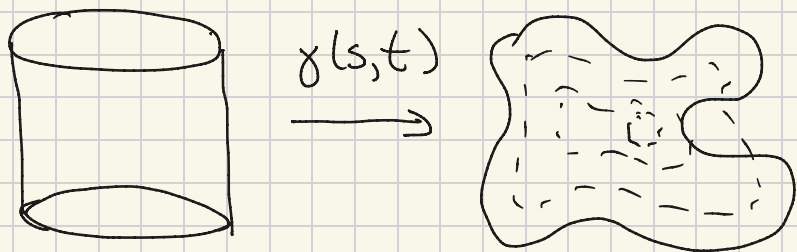
Note. When the tangent winds more than once around the circle, we have to be careful!



Proposition. If γ is a closed, embedded curve, then $\int_{\gamma} \chi(s) ds = 2\pi$.



Note. Actually a corollary of



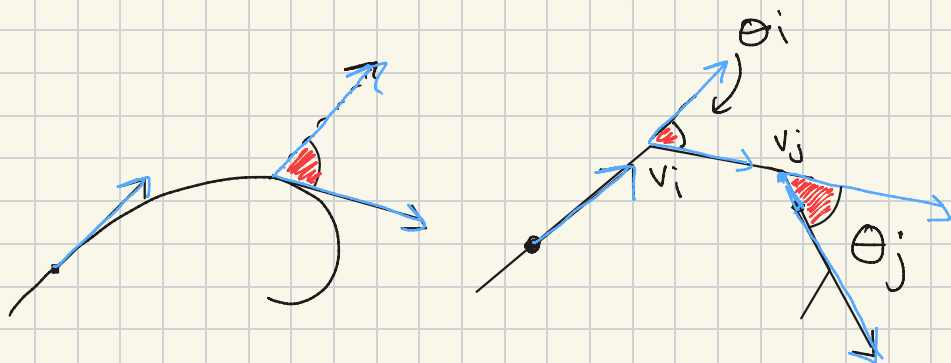
Theorem. If $\gamma(s, t): S^1 \times [0, 1]$ is a smooth map with $\gamma_s(s_0, t_0) \neq \vec{0}$ for all (s_0, t_0) , then the total curvature of $\gamma(s, 0)$ and $\gamma(s, 1)$ are equal.

Extension. (Planar Gauss-Bonnet)

If Ω is a domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and n holes,

$$\int_{\partial\Omega} \kappa(s) ds = 2\pi(n+1).$$

How do we generalize to polygons?



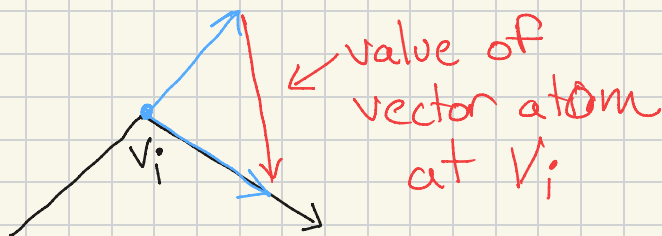
Definition 1. The discrete curvature at a vertex of a polygon is the turning angle. θ_i

Motivation. If $\int_a^b g(x) dx = G(b) - G(a)$ for all (a, b) , and G is differentiable almost everywhere then $g(x) = G'(x)$.

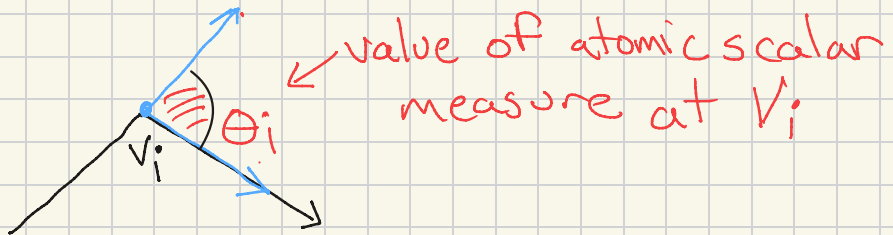
If there is some Radon measure μ so that $\int_a^b \mu = G(b) - G(a)$ for all (a, b) we say μ is the distributional derivative of G .

Idea 1. $X(s)ds$ is replaced by a measure with atoms at vertices, while $X(s)N(s)$ is a vector measure with atoms so

" $\frac{d}{ds}T(s) :=$ atomic^{vector} measure at corners.



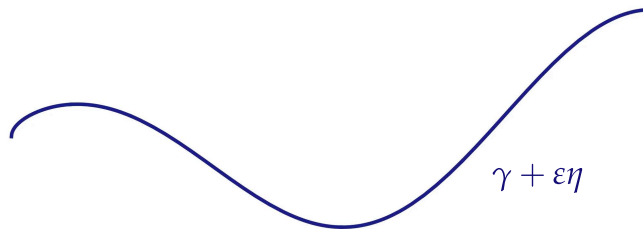
Note that " $\frac{d}{ds} \phi(s)$ " := atomic scalar
measure at corners



Second idea. Length variation.

Length Variation

- More formally, consider an *arbitrary* change in the curve γ , given by a function $\eta : [0, L] \rightarrow \mathbb{R}^2$ with $\eta(0) = \eta(L) = 0$.

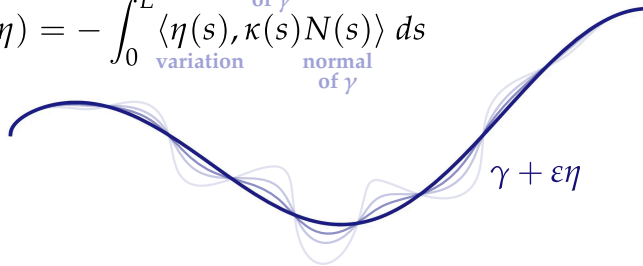


Length Variation

- More formally, consider an *arbitrary* change in the curve γ , given by a function $\eta : [0, L] \rightarrow \mathbb{R}^2$ with $\eta(0) = \eta(L) = 0$.
Then

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \text{length}(\gamma + \varepsilon\eta) = - \int_0^L \underbrace{\langle \eta(s), \kappa(s) N(s) \rangle}_{\text{variation}} ds$$

curvature of γ
normal of γ



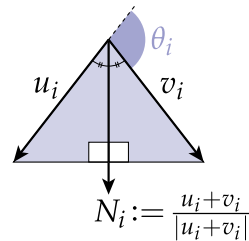
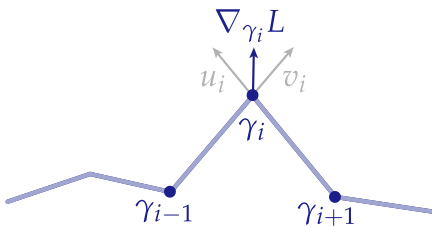
- Therefore, the motion that most quickly decreases length is $\eta = \kappa N$.

Prove it here (if time permits)

We can compute ∇Length as a function of the vertex positions.

Gradient of Length for a Discrete Curve

- To find the motion that most quickly increases the *total* length L , we now just sum the contributions of each segment:



- Using some simple trigonometry, we can also express the length gradient in terms of the exterior angle θ_i and the angle bisector N_i :

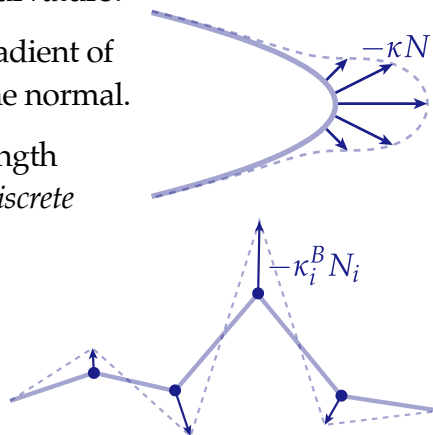
$$\nabla_{\gamma_i} L = 2 \sin(\theta_i/2) N_i$$

Discrete Curvature (Length Variation)

- How does this help us define discrete curvature?
- Recall that in the smooth setting, the gradient of length is equal to the curvature times the normal.
- Hence, our expression for the *discrete* length variation provides a definition for the *discrete* curvature times the *discrete* normal.

$$\kappa_i^B N_i := 2 \sin(\theta_i/2) N_i$$

(length variation)



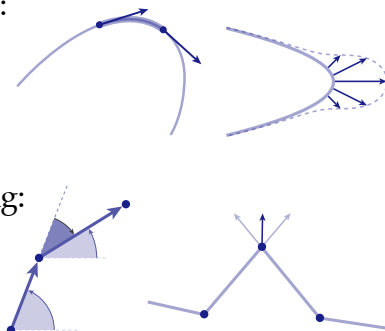
A Tale of Two Curvatures

- To recap what we've done so far: we considered two **equivalent** definitions in the smooth setting:

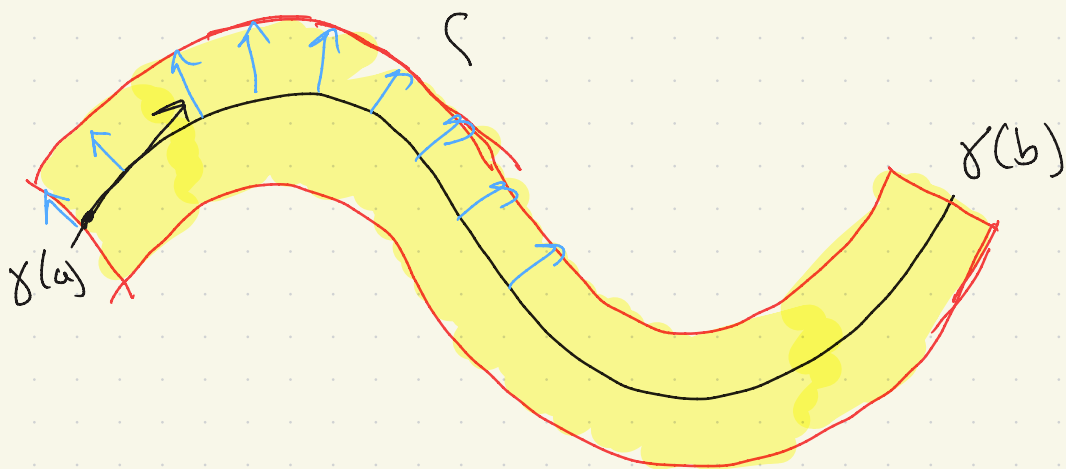
1. turning angle
2. length variation

- These perspectives led to two **inequivalent** definitions of curvature in the discrete setting:

1. $\kappa_i^A := \theta_i$
2. $\kappa_i^B := 2 \sin(\theta_i/2)$



- For *small* angles, both definitions agree ($\sin(\varepsilon) \approx \varepsilon$).
- Is one "better"? Are there more possibilities? Let's keep going...



Proposition. (Steiner's Tube Formula)

If $\gamma(s, t) := \gamma(s) + tN(s)$ then

$$\text{Len}(\gamma(s, t)) - \text{Len}(\gamma(s, 0)) = t \int_a^b K(s) ds.$$

Proof.

$$\frac{d}{ds} \gamma(s, t) = \gamma'(s) + tN'(s)$$

$$= T(s) - tKT(s)$$

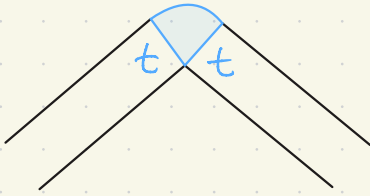
$$= (1 - tK)T(s)$$

So

$$\text{Len}(\gamma(s,t)) = \int_a^b \sqrt{1 - t^2 K(s)} ds$$

$$= \text{Len}(\gamma(s,0)) - t \int_a^b K(s) ds. \quad \square$$

We can offset polygon edges by t .

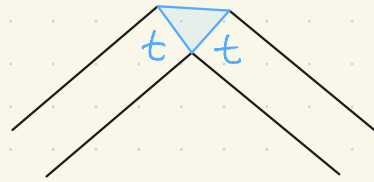


circle arc

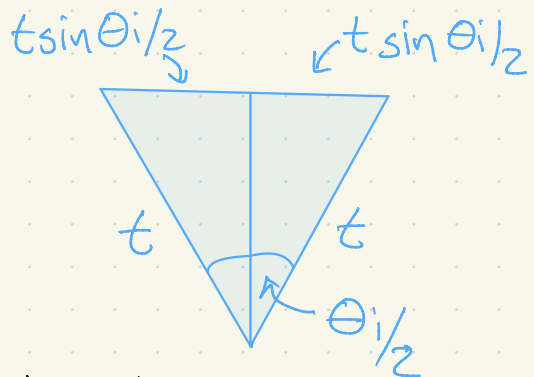
$$\text{Length} = t \theta_i$$

$$\text{curvature} = \theta_i$$

same as $\hat{\kappa}_1$



straight line



$$\text{Length} = 2t \sin \theta_i / 2$$

$$\text{curvature} = 2 \sin \theta_i / 2$$

same as $\hat{\kappa}_2$

The choice ∇ leading again to $\hat{X}_1$:

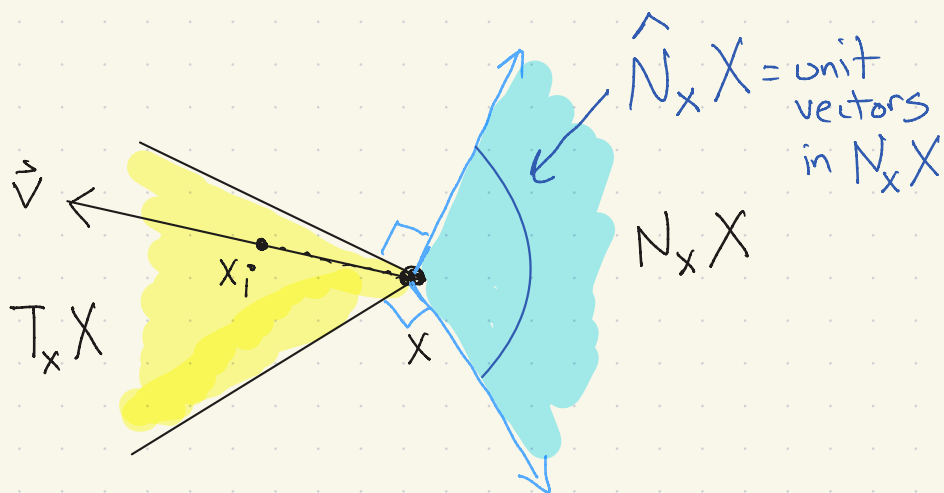
Definition. The tangent cone to a set A at x_0 is given by

$$T_{\vec{x}} X = \left\{ \vec{v} \in \mathbb{R}^n \mid \exists \vec{x}_i \rightarrow \vec{x}, \vec{x}_i \in X \right. \\ \left. c_i \rightarrow 0, c_i > 0 \right\}$$

$$\text{st } c_i(\vec{x}_i - \vec{x}) \rightarrow \vec{v}$$

The normal cone

$$N_{\vec{x}} X = \left\{ \vec{w} \in \mathbb{R}^n \mid \forall \vec{v} \in T_x X, \langle \vec{v}, \vec{w} \rangle \leq 0 \right\}.$$



Definition. If $X \subset \mathbb{R}^n$, then X_ϵ is

$$X_\epsilon = \{ \vec{y} \in \mathbb{R}^n \mid \exists \vec{x} \in X \text{ st. } \|\vec{y} - \vec{x}\| < \epsilon \}.$$

Definition. A subset X of $\mathbb{R}^n$ has reach ϵ if every element of X_ϵ has a unique nearest neighbor in X .

Theorem (Federer/Morgan Steiner formula).

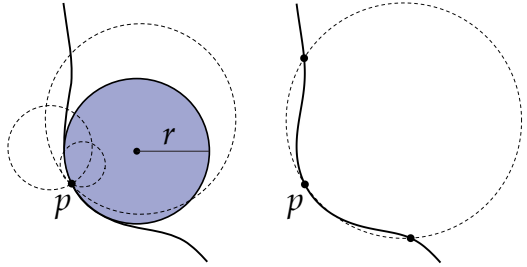
If X is a compact set of positive reach in $\mathbb{R}^2$, then

$$\mathcal{L}^2(X_\epsilon) = \mathcal{L}^2(X) + \epsilon \mathcal{H}^1(\partial X) + \epsilon^2 \int_{\hat{N}_X} w$$

where w measures length on S^1 ,
as long as $\epsilon < \text{reach}(X)$.

Osculating Circle

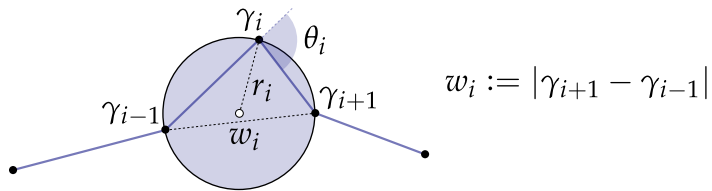
- One final idea is to consider the **osculating circle**, which is the circle that best approximates a curve at a point p



- More precisely, if we consider a circle passing through p and two equidistant neighbors to the “left” and “right” (resp.), the osculating circle is the limiting circle as these neighbors approach p .
- The curvature is then the reciprocal of the radius: $\kappa(p) = \frac{1}{r(p)}$

Discrete Curvature (Osculating Circle)

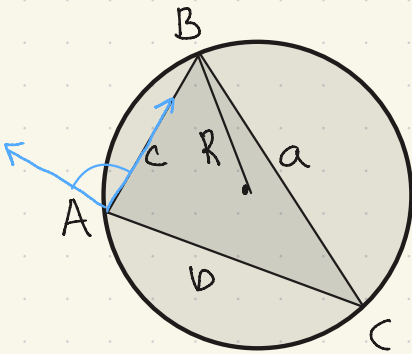
- A natural idea, then, is to consider the *circumcircle* passing through three consecutive vertices of a discrete curve:



- Our *fourth* discrete curvature is then the reciprocal of the radius:

$$\kappa_i^D := \frac{1}{r_i} = 2 \sin(\theta_i) / w_i$$

Proof. The law of sines



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

so setting $A = \pi - \theta_i$
and $a = \omega_i$, and
solving for $1/R$, we

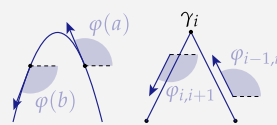
$$\text{get } \frac{\omega_i}{\sin(\pi - \theta_i)} = 2R$$

and $\frac{1}{R} = \frac{2 \sin \theta_i}{\omega_i}$, as claimed. $\square$

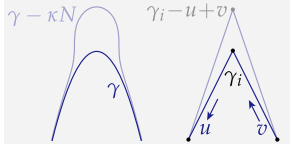
Playing the Game

- In the **smooth** setting, there are several other **equivalent** definitions of curvature.
- IDEA:** perhaps some of these definitions can be applied *directly* to our discrete curve!
- Actually, all four can—and will have different consequences...

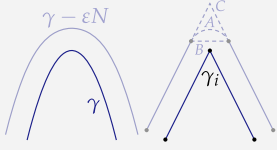
TURNING ANGLE



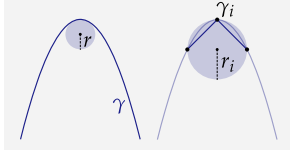
LENGTH VARIATION



STEINER FORMULA



OSCULATING CIRCLE



"Curve-shortening" flow.

Suppose we define a flow on plane curves by letting $F(x, t)$ be a solution to the PDE

$$F_t = \bar{K}^2 F + \kappa N \quad \bar{K}^2 = \frac{1}{2\pi} \int \kappa^2 ds.$$

where ds is integration wrt arclength and $F(x, 0) = \gamma(x)$ is a curve of length 2π , and N is the Frenet normal.

Lemma. The length of $F(x, t_0) \equiv 2\pi$.

Proof. The rhs is parametrization independent, so wlog assume that x is the arclength parameter.

Then

$$\frac{d}{dt} \text{Len}(F(x,t)) = \frac{d}{dt} \int \langle F_x, F_x \rangle^{1/2} dx$$

$$= \int \frac{1}{\cancel{2\langle F_x, F_x \rangle^{1/2}}} \cdot \cancel{2} \langle F_{xt}, F_x \rangle dx$$

$$= \int \langle F_{tx}, F_x \rangle dx$$

$$= \int \left\langle \frac{d}{dx} (\bar{x}^2 F + xN), F_x \right\rangle dx$$

$$= \int \langle \bar{x}^2 T + x_x N - x(xT), T \rangle dx$$

$$= \int \bar{x}^2 - x^2 dx = 0,$$

recalling that x is an arclength parameter on γ . $\square$

Note: If $\gamma(s) = F(s, 0)$ is the unit circle, the right hand side is $\vec{0}$.

$$\begin{aligned} F_t &= \bar{K}^2 F + \chi N \\ &= F + N = \begin{bmatrix} \cos x \\ \sin x \end{bmatrix} + \frac{d^2}{dx^2} \begin{bmatrix} \cos x \\ \sin x \end{bmatrix} = 0. \end{aligned}$$

Note: We know $\int \chi ds = 2\pi$ for any embedded closed curve, so by Jensen's inequality,

$$\bar{K}^2 = \frac{1}{2\pi} \int K^2 ds \geq \left(\frac{1}{2\pi} \int K ds \right)^2 = 1.$$

$$\underbrace{\left(\begin{array}{l} \text{(expectation)} \\ \text{of } K^2 \end{array} \right) \left(\begin{array}{l} \text{(square of} \\ \text{expectation)} \\ \text{of } K \end{array} \right)}_{\text{equality } \Leftrightarrow K^2 \equiv 1}$$

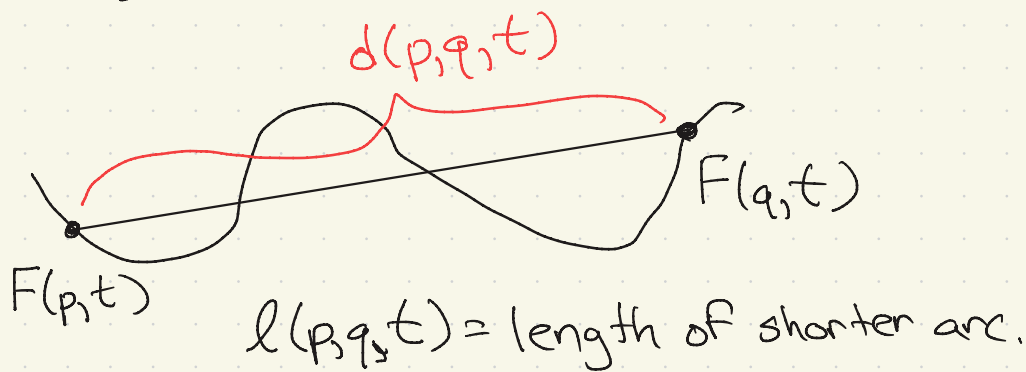
because $f(x) = x^2$ is strictly convex.

We want to show

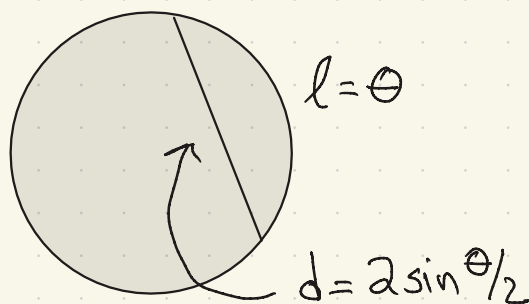
Theorem (Gage, Grayson, ...)

The flow converges to the unit circle,
all intermediate curves are smooth,
and if the initial curve γ is embedded,
so are all intermediate curves.

Chords and arcs.



For the unit
circle, we again
have the ratio.



Definition. Let

new x unrelated
to previous!
↓

$$f(x, t) = 2e^t \arctan(e^{-t} \sin(x/2)).$$

and abbreviate

$$z = e^{-t} \sin(x/2) \quad Q = 1 + z^2$$

$$c = \cos(x/2) \quad o = \sin(x/2) = e^t z.$$

Proposition. For each $t \in \mathbb{R}$, if $f' = f_x, f'' = f_{xx}$

$$(i) \quad f(0, t) = f(2\pi, t) = 0 \quad 0 < f(x, t) < 2\sin(x/2) < x \\ f(x, t) = f(2\pi - x, t)$$

$$(ii) \quad f' = \frac{\cos(x/2)}{1+z^2}, \quad f'' = \frac{-\sin(x/2)}{2(1+z^2)} - \frac{e^{-t} z \cos^2(x/2)}{(1+z^2)^2}$$

$$0 \leq f' \leq 1 \text{ on } [0, \pi] \quad f'' \text{ strictly concave on } [0, 2\pi]$$

$$f'(\pi) = 0$$

$$(iii) \quad \text{If we let } g(z) := \arctan z - \frac{z}{1+z^2},$$

$$f_t = 2e^t g(z), \quad g(0) = 0, \quad g'(z) = \frac{2z^2}{(1+z^2)^2} > 0$$

$$\text{and } \lim_{t \rightarrow \infty} f(x, t) = 2\sin(x/2), \quad \lim_{t \rightarrow -\infty} f(x, t) = 0.$$

(iv) (strict subadditivity) If $x, y > 0$,
and $x+y \leq 2\pi$, then

$$f(x+y, t) < f(x, t) + f(y, t)$$

(v) (Taylor expansion) For $a > 0$, $x \rightarrow 0$

$$f(x, -\log a) = x - \frac{1+2a^2}{24} x^3 + O(x^5)$$

(vi) $f - xf' > 0$ for $x \in (0, 2\pi)$.

Proof of (ii). Looking back at our definitions, we have $z = e^{-t}\sigma$

$$z' = e^{-t} \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2} = \frac{1}{2} e^{-t} c$$

and

$$f = 2e^t \arctan(z)$$

so

$$f' = 2e^t \cdot \frac{1}{1+z^2} z' = \frac{c}{Q}$$

Now

$$c = \cos(x/2) \Rightarrow c' = -\sin(x/2) \cdot 1/2 = -\sigma/2$$

while

$$Q = 1 + z^2 \Rightarrow Q' = 2zz' = z \cdot e^{-t} c$$

Now

$$f'' = \frac{c'}{Q} - \frac{cQ'}{Q^2} = \frac{-\sigma}{2Q} - \frac{ze^{-t}c^2}{Q^2}.$$

Proof of (iii). Note $z_t = -z$.

$$f_t = 2e^t \arctan(z) + 2e^t \frac{z_t}{1+z^2}$$

$$= 2e^t \arctan(z) - 2e^t \frac{z}{1+z^2}$$

$$= 2e^t g(z).$$

Now

as a function of z ,

$$g'(z) = \frac{1}{1+z^2} - \left(\frac{1+z^2 - z(2z)}{(1+z^2)^2} \right)$$

$$= \frac{(1+z^2) - (1+z^2) + 2z^2}{(1+z^2)^2}$$

$$= \frac{2z^2}{(1+z^2)^2} > 0$$

and $g(0) = 0 - 0/1 = 0$, so $g(z) > 0$
for all z . 

Note that this means that

$f(l, t)$ increases monotonically
in t from 0 to $2\sin(l/2)$.

Definition. For any fixed time curve, let l and d be its length and chordlength functions and let

$$a(p, q) = \begin{cases} 0, & \text{if } d(p, q) \geq 2 \sin(l/2) \\ e^{\sigma_0}, & \text{if } 0 < d(p, q) < 2 \sin(l/2) \end{cases}$$

where we define σ_0 by $d = f(l, -\sigma_0)$ and assume $p \neq q$.

Note that for any $b > 0$, fixing (p, q)

$$a(p, q) \leq b \Rightarrow a(p, q) = 0 \text{ or } a(p, q) = e^{\sigma_0}$$

In the first case,

$$d \geq 2 \sin(l/2) > f(l, t) \text{ for any } t$$

In the second,

$$e^{\sigma_0} \leq b \Rightarrow \sigma_0 \leq \log b \Rightarrow -\sigma_0 \geq -\log b$$

but $f(\ell, -)$ is increasing in the second argument, so

$$\Rightarrow f(\ell, -\sigma_0) \geq f(\ell, -\log b)$$

$$\Rightarrow d \geq f(\ell, -\log b).$$

The converse is similar.

Key Lemma. $a(p, q)$ extends continuously to the diagonal with

$$a(p, p) = \sqrt{\frac{\max(K(p)^2 - 1, 0)}{2}}.$$

In particular $\bar{a} = \sup a(p, q) < \infty$.

Corollary. With $\bar{t} = \log \bar{a}$, at $t=0$

$$d(p, q) \geq f(\ell(p, q), -\bar{t})$$

This lets us finally state our main tool.

(Andrews-Bryan Comparison.) For any $t > 0$ and all $p, q \in S^1$, the chordlength and arclength functions $d(p, q, t)$ and $\ell(p, q, t)$ obey

$$d(p, q, t) \geq f(\ell(p, q, t), t - \bar{t})$$

Note that if $\ell(p, q) > 0$, so is the right-hand side: in a quantitative sense the flow preserves embeddedness!

Proof. We let $K \geq |K(p,t)|$ on $[0, t_1]$,

$$Z_\lambda(p, q, t) := d(p, q, t) - \lambda f(\ell(p, q, t), t - \bar{t})$$

where $\lambda \in (0, 1)$, $(p, t) \in S^1 \times [0, t_1]$.

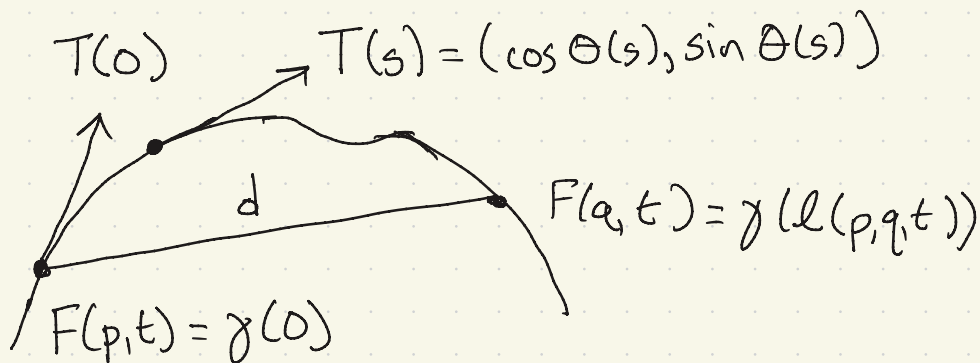
Lemma. (Lower bound on Z_λ near diagonal)

Let $\ell_0 := \min(\pi/K, \sqrt{3(1-\lambda)}K, \pi/2)$, then for $t \in [0, t_1]$, p, q with $\ell(p, q) \leq \ell_0$

$$Z_\lambda(p, q, t) \geq \frac{1}{2}(1-\lambda)\ell(p, q, t) > 0$$

Proof. Step 1.

Let γ be a unit-speed reparametrization of the portion of $F(s, t)$ between p and q .



The curvature of γ is $\leq K$, so
 $|\theta(s) - \theta(0)| < Ks$ along this arc.

(Since $s < l_0 \leq \pi/K$, this angle is $< \pi$.)

Now

$$\begin{aligned} d &= \|F(q, t) - F(p, t)\| \\ &= \|F(q, t) - F(p, t)\| \|T(0)\| \\ &\geq \langle F(q, t) - F(p, t), T(0) \rangle \\ &= \langle \gamma(l) - \gamma(0), \gamma'(0) \rangle \\ &= \left\langle \int_0^l \gamma'(s) ds, \gamma'(0) \right\rangle \\ &= \int_0^l \langle \gamma'(s), \gamma'(0) \rangle ds \\ &= \int_0^l \cos(\theta(s) - \theta(0)) ds \end{aligned}$$

$$= \int_0^l \cos(|\theta(s) - \theta(0)|) ds$$

cos is even

But cos is decreasing on $[0, \pi]$,
and $|\theta(s) - \theta(0)| \leq Ks$ on $(0, l)$,

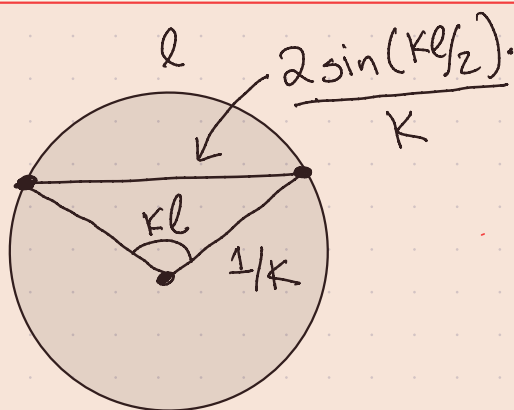
so

$$\geq \int_0^l \cos(Ks) ds$$

$$= \frac{\sin(Kl)}{K}$$

$$\geq l - \frac{K^3 l^3}{\frac{3!}{K}}$$

$$= l - \frac{K^2 l^3}{6}$$



Note: Could be improved to $2 \sin(Kl/2)$ which is $\geq \sin(Kl)$.

Now by your homework, $f(l, t) \leq l$, so

$$\begin{aligned} Z_\lambda &\geq d - \lambda l \geq l - \lambda l - \frac{\kappa^2 l^3}{6} \\ &= (1 - \lambda) \left(l - \frac{\kappa^2 l^3}{6(1 - \lambda)} \right) \\ &= (1 - \lambda) l \left(1 - \frac{\kappa^2 l^2}{6(1 - \lambda)} \right) \end{aligned}$$

Now $l \leq l_0 \leq \sqrt{3(1 - \lambda)}/\kappa$ by construction,
so

$$1 - \frac{\kappa^2 l^2}{6(1 - \lambda)} \geq 1 - \frac{\cancel{\kappa^2}}{\cancel{6(1 - \lambda)}} \cdot \frac{\cancel{3(1 - \lambda)}}{\cancel{\kappa^2}} \geq 1/2$$

2

and we have

$$Z_\lambda \geq \frac{1}{2} (1 - \lambda) l > 0. \quad \square$$

Big Picture. Curvature bound lets us control behavior near the diagonal (part of the boundary).

So consider

$$M(t) := \min \{ Z_\lambda(p, q, t) \mid p, q \text{ have } l(p, q) \geq l_0 \}$$

when $t=0$, we know (corollary) that

$$d(p, q) \geq f(l(p, q), -\bar{t})$$

so

$$\begin{aligned} Z_\lambda(p, q, 0) &= d(p, q, 0) - \lambda f(l(p, q, 0), -\bar{t}) \\ &\geq (1-\lambda) f(l, -\bar{t}) \end{aligned}$$

Now f is increasing in l on $[0, \pi]$ and symmetric around π on $[0, 2\pi]$, so its min $\downarrow$ on $[l_0, 2\pi - l_0]$ is at l_0
in t

but we know $f(l_0, -\bar{\epsilon}) > 0$, so

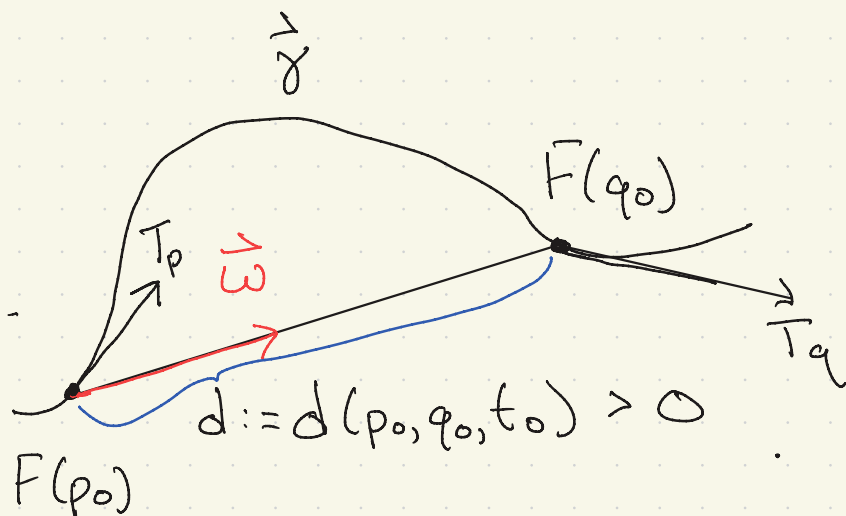
$$Z_\lambda(p, q, 0) > 0 \quad \text{when } l(p, q) \geq l_0.$$

So we have $M(0) > 0$. Now if $M(t) > 0$ for all $t \in [0, t_1]$, then we're done.

The rest of the proof is devoted to showing that the other case is impossible.

Let t_0 be the first time $M=0$ and suppose $Z_\lambda(p_0, q_0, t_0) = 0$. We know (barrier lemma) $l(p_0, q_0, t_0) > l_0$. Further, since $Z_\lambda(p, q, t) > 0$ for $t < t_0$,

$$\frac{\partial}{\partial t} Z_\lambda(p_0, q_0, t_0) \leq 0 \quad (\star)$$



Notice that $F(-, t_0)$ is still embedded (we hit the barrier function). Let

$$\vec{\omega} = \frac{\vec{F}(q_0) - \vec{F}(p_0)}{d}, \quad \vec{T}_p = \vec{T}(p_0), \quad T_q = \vec{T}(q_0).$$

Now (p_0, q_0) is an interior min of Z_λ , so

$$0 = \frac{\partial}{\partial p} Z_\lambda(p, q, t_0) = \frac{\partial}{\partial p} d - \lambda f' \frac{\partial \ell}{\partial p}$$

$\uparrow -1$

$$= -\langle \vec{\omega}, T_p \rangle + \lambda f'$$

$$0 = \frac{\partial}{\partial q} Z_\lambda(p, q, t_0) = \langle \omega, T_q \rangle - \lambda f'$$

$\uparrow \frac{\partial \ell}{\partial q} = +1$

which means that

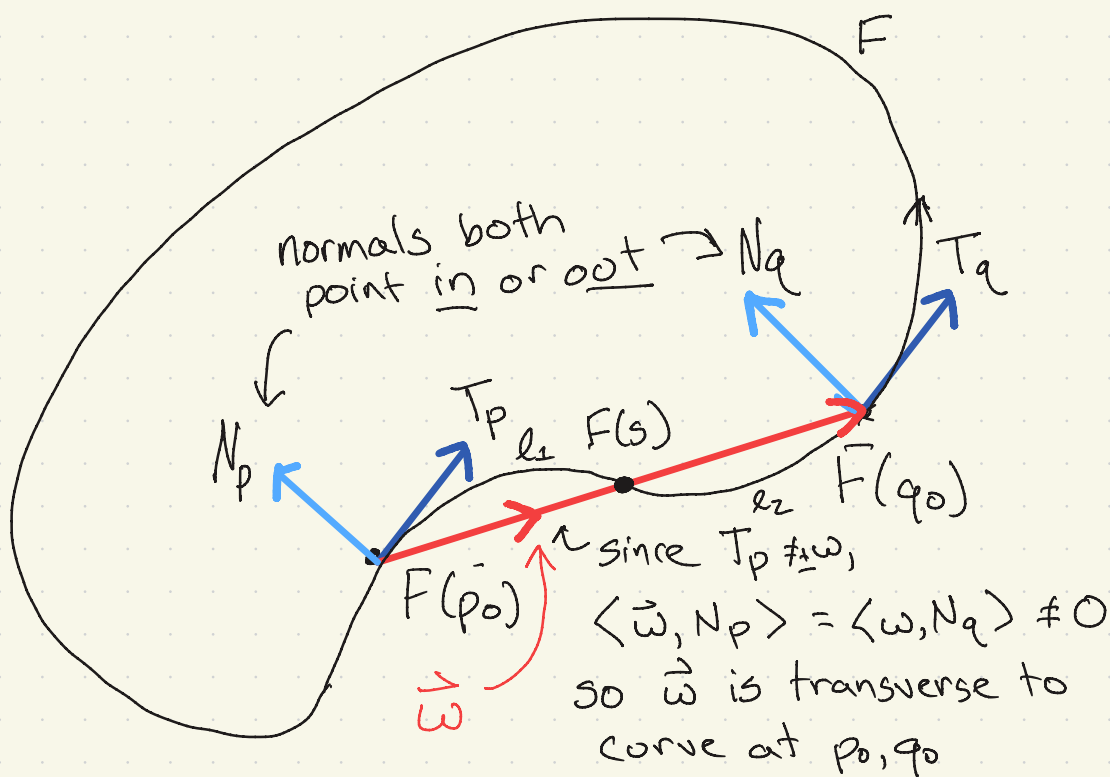
$$\langle \vec{\omega}, T_p \rangle = \langle \vec{\omega}, T_q \rangle = \lambda f' =: c_0$$

where we know $0 < f' < 1$, so $c_0 \in [0, \lambda]$.

Note that $\lambda < 1 \Leftrightarrow T_p, T_q \neq \vec{\omega}$, and

$\lambda > 0 \Rightarrow T_p, T_q \neq -\vec{\omega}$.

Case 1. Parallel tangents.



We see that the chord from $F(p_0)$ to $F(q_0)$ meets γ at some interior point $F(s)$. Wlog, we have $p_0 < s < q_0$ so

$$l_1 = s - p_0 \quad l_2 = q_0 - s$$

where these distances are measured along the arc. Now

$$l(p_0, q_0) = q_0 - p_0 \text{ or } 2\pi - (q_0 - p_0)$$

while $l_1 + l_2 = q_0 - p_0$. In either case,

$$f(l_1 + l_2) = f(l(p_0, q_0))$$

since $f(x) = f(2\pi - x)$. Similarly,

$$l_1 = l(p_0, s) \text{ or } l_1 = 2\pi - l(p_0, s)$$

$$l_2 = l(s, q_0) \text{ or } l_2 = 2\pi - l(s, q_0)$$

but either way

$$f(l_1) = f(l(p_0, s)) \quad f(l_2) = f(l(s, q_0))$$

By colinearity of $F(p_0), F(s), F(q_0)$,

$$d(p_0, s) + d(s, q_0) = d(p_0, q_0).$$

Now

$$\begin{aligned} 0 &= Z_\lambda(p_0, q_0) = d(p_0, q_0) - \lambda f(\ell(p_0, q_0)) \\ &= d(p_0, s) + d(s, q_0) - \lambda f(\ell_1 + \ell_2) \end{aligned}$$

which by $f(x+y) \leq f(x) + f(y)$ (homework)

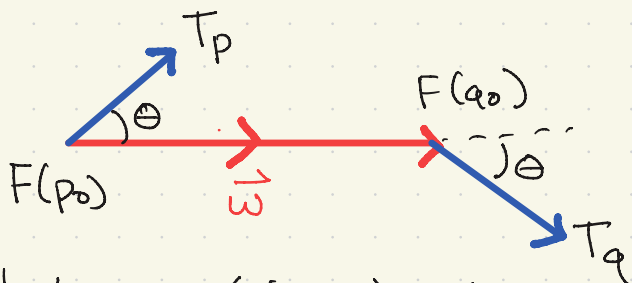
$$\geq d(p_0, s) - \lambda f(\ell_1) + d(s, q_0) - \lambda f(\ell_2)$$

$$= \underbrace{d(p_0, s) - \lambda f(\ell(p_0, s))}_{Z_\lambda(p_0, s)} + \underbrace{d(s, q_0) - \lambda f(\ell(s, q_0))}_{Z_\lambda(s, q_0)}$$

$$\geq 0$$

because Z_λ is ≥ 0 for all pairs at this time. ~~XX~~

Case 2. Opposite tangents.



We still have $\langle \vec{w}, T_p \rangle = \langle \vec{w}, T_q \rangle = \lambda F' = \cos \Theta$.

Now our mission is to compute the Hessian of Z_λ in the arclength variables p and q , which is positive semidefinite b/c (p_0, q_0) is a global min in (p, q) .

$$d(p, q) = \langle F(q) - F(p), F(q) - F(p) \rangle^{1/2}$$

$$\frac{\partial}{\partial p} d(p, q) = \frac{\langle T(p), F(p) - F(q) \rangle}{\langle F(q) - F(p), F(q) - F(p) \rangle^{1/2}}$$

Side computations,

$$\begin{aligned} \frac{\partial}{\partial p} \langle T(p), F(p) - F(q) \rangle &= \langle K_p N(p), F(p) - F(q) \rangle \\ &\quad + \langle T(p), T(p) \rangle \\ &= \langle K_p N(p), F(p) - F(q) \rangle + 1 \end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial q} \langle T(p), F(p) - F(q) \rangle &= - \langle T(p), T(q) \rangle \\ &= - \cos(2\theta) = 1 - 2\cos^2\theta = 1 - 2c_0^2\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial p} \frac{1}{\langle F(p) - F(q), F(p) - F(q) \rangle^{1/2}} \\ &= - \frac{1}{2} \cdot \frac{2 \langle T(p), F(p) - F(q) \rangle}{\langle F(p) - F(q), F(p) - F(q) \rangle^{3/2}} = \\ &= \frac{- \langle T(p), F(p) - F(q) \rangle}{d^3}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial q} \frac{1}{\langle F(p) - F(q), F(p) - F(q) \rangle^{1/2}} \\ &= - \frac{1}{2} \cdot \frac{2 \langle T(q), F(q) - F(p) \rangle}{\langle F(p) - F(q), F(p) - F(q) \rangle^{3/2}} = \\ &= \frac{- \langle T(q), F(q) - F(p) \rangle}{d^3}\end{aligned}$$

These allow us to assemble

$$\begin{aligned} \frac{\partial^2}{\partial p^2} d &= \frac{1}{d} \left(\langle K_p N_p, F(p) - F(q) \rangle + 1 \right) \\ &+ \frac{\langle T(p), F(p) - F(q) \rangle \left(- \langle T(p), F(p) - F(q) \rangle \right)}{d^3} \\ &= - \langle K_p N_p, w \rangle + \frac{1}{d} - \frac{1}{d} \langle T(p), w \rangle^2 \end{aligned}$$

and $= - \langle K_p N_p, w \rangle + \frac{1}{d} (1 - c_0^2).$

$$\frac{\partial^2}{\partial q \partial p} d = \frac{1 - 2c_0^2}{d} +$$

$$\frac{\langle T(p), F(p) - F(q) \rangle \langle -T(q), F(q) - F(p) \rangle}{d^3}$$

$$= \frac{1}{d} (1 - 2c_0^2 + \langle T(p), w \rangle \langle T(q), w \rangle)$$

$$= \frac{1}{d} (1 - 2c_0^2 + c_0^2) = \frac{1}{d} (1 - c_0^2).$$

and by almost - symmetry

$$\frac{\partial^2}{\partial q^2} d = + \langle K_q N(q), \omega \rangle + \frac{1}{d} (1 - c_0^2).$$

↑ because

$$\begin{aligned} \frac{\partial}{\partial q} \langle T(q), F(q) - F(p) \rangle \\ = \langle x_q N_q, F(q) - F(p) \rangle + 1 \end{aligned}$$

Now we have

+ d\omega

$$Hd = \begin{bmatrix} -\langle x_p N_p, \omega \rangle & 0 \\ 0 & + \langle x_q N_q, \omega \rangle \end{bmatrix} + \frac{1}{d} (1 - c_0^2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

So computing

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

we see that

$$\begin{bmatrix} 1 & -1 \end{bmatrix} H_d \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -\langle X_p N_p, \omega \rangle \\ -\langle X_q N_q, \omega \rangle \end{bmatrix}$$

$$= \langle X_q N_q - X_p N_p, \omega \rangle.$$

Now in our expression for Z_λ we also have $f(l(p, q))$, so

$$\frac{\partial}{\partial p} f(l(p, q)) = f'(l(p, q)) l_p$$

$$\frac{\partial}{\partial q} f(l(p, q)) = f'(l(p, q)) l_q$$

$$\frac{\partial^2}{\partial p^2} f(l(p, q)) = f''(l(p, q)) l_p^2$$

$$\frac{\partial}{\partial q \partial p} f(l(p, q)) = f''(l(p, q)) l_p l_q$$

$$\frac{\partial^2}{\partial q^2} f(l(p, q)) = f''(l(p, q)) l_q^2$$

Now whether $l(p, q) = q - p$ or $2\pi - (q - p)$,
we have $|l_p| = |l_q| = 1$ and $l_p l_q = -1$.

Thus

$$Hf = f'' \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and for our variation, we just
compute

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 4$$

and we finally get the second variation

$$\begin{bmatrix} -1 & 1 \end{bmatrix} HZ_\lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \langle K_q N_q - K_p N_p, \omega \rangle - 4\lambda f'' \geq 0.$$

The endgame begins here...

Recall that we know

$$\langle \cdot | \cdot \rangle \geq \frac{\partial}{\partial t} Z_\lambda$$

So let's compute

$$\begin{aligned} \frac{\partial}{\partial t} d &= \frac{\partial}{\partial t} \langle F(q) - F(p), F(q) - F(p) \rangle^{1/2} \\ &= \frac{1}{2d} \langle F(q) - F(p), \frac{\partial}{\partial t} F(q) - \frac{\partial}{\partial t} F(p) \rangle \\ &= \frac{1}{d} \langle F(q) - F(p), \bar{X}^2 F(q) - \bar{X}^2 F(p) \rangle \\ &\quad + \frac{1}{d} \langle F(q) - F(p), X_q N_q - X_p N_p \rangle \\ &= \bar{X}^2 d + \langle \omega, X_q N_q - X_p N_p \rangle \end{aligned}$$

and

$$\partial_t l = \int_p^q \bar{x}^2 - x^2 dx$$

$$= l \bar{x}^2 - \int_p^q x^2(s) ds.$$

by our original proof that length is preserved during the flow. Now as a function of p, q, t

$$\frac{\partial}{\partial t} Z_\lambda = \frac{\partial}{\partial t} d - \lambda \frac{\partial}{\partial t} \cdot f(l(p, q), t)$$

$$= \bar{x}^2 d + \langle \omega, K_q N_q - K_p N_p \rangle$$

$$- \lambda f'(l, t) \cdot \frac{\partial}{\partial t} l - \lambda \frac{\partial}{\partial t} f$$

$$= \bar{x}^2 d + \langle \omega, K_q N_q - K_p N_p \rangle$$

$$- \lambda f' \left(l \bar{x}^2 - \int_p^q x^2 ds \right) - \lambda f_t.$$

!! part
of 2nd
variation

$$\geq 4\lambda f'' + \bar{\kappa}^2(d - \lambda f'l) + \lambda f' \int_p^q \kappa^2 ds - \lambda f_t$$

↑ our second variation showed

$$\langle \omega, \kappa_q N_q - \kappa_p N_p \rangle \geq 4\lambda f''$$

Now $Z_\lambda = 0$ at this place and time,

so $d = \lambda f$, so

$$= \lambda \left(4f'' + \bar{\kappa}^2(f - f'l) + f' \int_p^q \kappa^2 ds - f_t \right)$$

We know $\bar{\kappa}^2 \geq 1$ and $\underbrace{f - f'l}_{\text{homework}} > 0$

$$\text{so } \bar{\kappa}^2(f - f'l) = \underbrace{(\bar{\kappa}^2 - 1)(f - f'l)}_{\geq 0} + \underbrace{(f - f'l)}_{> 0}.$$

By Jensen's inequality $\overset{\geq 0}{\text{as before}}$

$$\frac{1}{l} \int_p^q \kappa^2 ds \geq \left(\frac{1}{l} \int_p^q \kappa ds \right)^2 = \frac{(2\theta)^2}{l^2}$$

↑ the change in tangents is $2\theta < \pi$
b/c both have positive dot with ω

Now $\cos \Theta = \lambda f'$, so $\Theta^2 = \arccos^2(\lambda f')$.

Homework. If $h(z) = (\arccos(z))^2$,

h is decreasing and convex (concave up) on $[0, 1]$. So

$$\Theta^2 = h(\lambda f') \geq h(f')$$

$\nwarrow \lambda < 1$
 $f' > \lambda f'$
 h decreasing

So

$$\int_p^q X^2 ds \geq \frac{4}{\ell} h(f')$$

and since $f' > 0$ (homework) we have

$$f' \int_p^q X^2 ds \geq \frac{4f'}{\ell} h(f')$$

Now h is convex, so it lies above the tangent line at $z = \cos(\ell/2)$.

At this point

$$h(z) = \arccos(\cos(\ell/2))^2 = \ell^2/4$$

while

$$h'(z) = \frac{-2 \arccos(z)}{\sqrt{1-z^2}} = \frac{-\cancel{2} \cdot \cancel{\ell/2}}{\sin(\ell/2)}$$

so the line is

$$y(z) = -\frac{\ell}{\sin(\ell/2)} (z - \cos(\ell/2)) + \ell^2/4$$

and

$$h(f') \geq \ell^2/4 - \frac{\ell}{\sin(\ell/2)} (f' - \cos(\ell/2)).$$

Thus

$$f' \int_P^q K^2 ds \geq \frac{4f'}{\ell} h(f')$$

$$\geq f' \ell - \frac{4f'}{\sin(\ell/2)} (f' - \cos(\ell/2)).$$

Now we go back to

$$= \lambda (4f'' + \bar{\kappa}^2(f - f'\ell) + f' \int_P^q \kappa^2 ds - f_\epsilon)$$

and put in everything we've learned.
The blue terms become

$$\square \geq \underbrace{(\bar{\kappa}^2 - 1)(f - f'\ell)}_{\geq 0} + \underbrace{(f - \cancel{f'\ell})}_{> 0}.$$

$$\cancel{f'\ell} - \frac{4f'}{\sin(\ell/2)} (f' - \cos(\ell/2)).$$

$$\geq (\bar{\kappa}^2 - 1)(f - f'\ell) + f - \frac{4f'}{\sin(\ell/2)} (f' - \cos(\ell/2)).$$

So we finally have

$$0 \geq \lambda ((\bar{\kappa}^2 - 1)(f - f'\ell) + \tilde{\mathcal{L}}f)$$

Where $\tilde{\mathcal{L}}$ is the 2nd order nonlinear differential operator

$$\tilde{\mathcal{L}}(f) := 4f_{\ell\ell} + f - \frac{4f_{\ell}}{\sin(\ell/2)} (f_{\ell} - \cos(\ell/2)) - \partial_t f.$$

Miracle Homework. $\tilde{\mathcal{L}}(f) \equiv 0$.

So

$$0 \geq (\bar{\chi}^2 - 1)(f - f'\ell) \geq 0$$

We conclude that $\bar{\chi}^2 = 1$ (remember, $f - f'\ell > 0$), and hence $\chi \equiv 1$ and the curve is a circle. But then

$$d = 2\sin(\ell/2) > f(\ell), \text{ so}$$

$$0 = \mathbb{Z}_{\lambda} = d - \lambda f > (1 - \lambda)f > 0$$

(At last!) a contradiction. to case 2.

This completes the proof. 

We now prove convergence to the circle. Recall:

Definition. For any fixed time curve, let l and d be its length and chordlength functions and let

$$a(p, q) = \begin{cases} 0, & \text{if } d(p, q) \geq 2\sin(l/2) \\ e^{\sigma_0}, & \text{if } 0 < d(p, q) < 2\sin(l/2) \end{cases}$$

where we define σ_0 by $d = f(l, -\sigma_0)$ and assume $p \neq q$.

Since $d \geq f(l, t - \bar{t}) = f(l, -\log e^{\bar{t}-t})$

we have $f(l, -\sigma_0) \geq f(l, -\log e^{\bar{t}-t})$

and (since f increasing in t),

$$-\sigma_0 \geq -\log e^{\bar{t}-t} \Rightarrow e^{\sigma_0} \leq e^{\bar{t}-t}$$

Thus

$$a(p, q, t) \leq e^{\bar{t}-t}$$

for all p, q and t . letting $q \rightarrow p$,

$$\sqrt{\frac{\max \{K_p^2 - 1, 0\}}{2}} \leq e^{\bar{t}-t}$$

or

$$\max \{K_p^2 - 1, 0\} \leq 2e^{-2(t-\bar{t})}$$

Now $\bar{K}^2 \geq 1$, so $K_p^2 \geq 1$ for some p
and $\max_p K_p^2 \geq 1$. At that p ,

$$\max_p K_p^2 \leq 1 + 2e^{-2(t-\bar{t})}$$

and we see max curvature decays
to 1 exponentially fast!

Corollary. $\text{Var}(X) \leq 2e^{-2(t-\bar{t})}$

Since $\frac{1}{2\pi} \int X ds \equiv 1$ (embeddedness),

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - 1 \\ &= \mathbb{E}(X^2 - 1).\end{aligned}$$

But we just showed that

$$\begin{aligned}2e^{-2(t-\bar{t})} &\geq \max(X^2 - 1) \\ &\geq \mathbb{E}(X^2 - 1).\end{aligned}$$

Now what happens to all these results for discrete flows!