



Definition. Let $e_1, \dots, e_n$ be the basis for an n -dimensional inner product space V . For each integer $0 \leq k \leq n$, let $\wedge^k$ denote an $\binom{n}{k}$ -dimensional vector space with basis elements denoted by $e_{i_1} \wedge \dots \wedge e_{i_k}$ for all possible sequences of indices $1 \leq i_1 < \dots < i_k \leq n$, corresponding to all possible "axis-aligned" k -dimensional volumes. Elements of $\wedge^k$ are called k -vectors. The wedge product is a bilinear map

$$\wedge_{k,l} : \wedge^k \times \wedge^l \rightarrow \wedge^{k+l}$$

uniquely determined by its action on basis elements; in particular, for any collection of *distinct* indices $i_1, \dots, i_{k+l}$,

$$(e_{i_1} \wedge \dots \wedge e_{i_k}) \wedge_{k,l} (e_{i_{k+1}} \wedge \dots \wedge e_{i_{k+l}}) := \text{sgn}(\sigma) e_{\sigma(i_1)} \wedge \dots \wedge e_{\sigma(i_{k+l})},$$

where σ is a permutation that puts the indices of the two arguments in canonical (lexicographic) order. Arguments with repeated indices are mapped to $0 \in \wedge^{k+l}$. For brevity, one typically drops the subscript on $\wedge_{k,l}$. Finally, the *Hodge star* on k -vectors is a linear isomorphism

$$\star : \wedge^k \rightarrow \wedge^{n-k}$$

uniquely determined by the relationship

$$\det(\alpha \wedge \star \alpha) = 1$$

where α is any k -vector of the form $\alpha = e_{i_1} \wedge \dots \wedge e_{i_k}$ and $\det$ denotes the determinant of the constituent 1-vectors (treated as column vectors) with respect to the inner product on V . The collection of vector spaces $\wedge^k$ together with the maps $\wedge$ and $\star$ define an *exterior algebra* on V , sometimes known as a *graded algebra*.

We start with some exterior algebra.

Definition. Suppose $\vec{e}_1, \dots, \vec{e}_n$ is a ^{formal} basis for an n -dimensional real vector space V .

$\wedge^k(V)$ = the $\binom{n}{k}$ -dimensional real vector space whose (formal) basis consists of vectors denoted

$$\vec{e}_{i_1} \wedge \dots \wedge \vec{e}_{i_k}$$

where $1 \leq i_1 < \dots < i_k \leq n$.

Elements of $\Lambda^k(V)$ are called k -vectors.

Example.

$$1\vec{e}_1 - 1\vec{e}_2 + 2\vec{e}_3 \in \Lambda^1(\mathbb{R}^3)$$

$$2\vec{e}_1 + 1\vec{e}_2 + 1\vec{e}_3 \in \Lambda^1(\mathbb{R}^3)$$

$$-3\vec{e}_2 \wedge \vec{e}_3 - 3\vec{e}_1 \wedge \vec{e}_3 + 3\vec{e}_1 \wedge \vec{e}_2 \in \Lambda^2(\mathbb{R}^3)$$

Definition. The wedge product is the bilinear map $\wedge: \Lambda^k \times \Lambda^l \rightarrow \Lambda^{k+l}$ defined uniquely by

$$(\vec{e}_{i_1} \wedge \dots \wedge \vec{e}_{i_k}) \wedge (\vec{e}_{i_{k+1}} \wedge \dots \wedge \vec{e}_{i_{k+l}})$$

$$= \text{sgn}(\sigma) \vec{e}_{i_{\sigma(1)}} \wedge \dots \wedge \vec{e}_{i_{\sigma(k+l)}}$$

where σ is the unique permutation so that these are in canonical order, and any input k -vectors which have indices in common map to zero.

Example.

$$(\vec{e}_3) \wedge (\vec{e}_1 \wedge \vec{e}_2) = \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \in \mathcal{L}^3(\mathbb{R}^3)$$

$$(\vec{e}_2) \wedge (\vec{e}_1) = -\vec{e}_1 \wedge \vec{e}_2 \in \mathcal{L}^2(\mathbb{R}^3)$$

$$(\vec{e}_1) \wedge (\vec{e}_2) = \vec{e}_1 \wedge \vec{e}_2 \in \mathcal{L}^2(\mathbb{R}^3)$$

$$(1\vec{e}_1 - 1\vec{e}_2 + 2\vec{e}_3) \wedge (2\vec{e}_1 + 1\vec{e}_2 + 1\vec{e}_3)$$

$$\begin{aligned}
 &= \cancel{2\vec{e}_1 \wedge \vec{e}_1}^0 + 1\vec{e}_1 \wedge \vec{e}_2 + 1\vec{e}_1 \wedge \vec{e}_3 \\
 &\quad + 2\vec{e}_1 \wedge \vec{e}_2 - \cancel{1\vec{e}_2 \wedge \vec{e}_2}^0 - 1\vec{e}_2 \wedge \vec{e}_3 \\
 &\quad - 4\vec{e}_1 \wedge \vec{e}_3 - 2\vec{e}_2 \wedge \vec{e}_3 - \cancel{2\vec{e}_3 \wedge \vec{e}_3}^0 \\
 &= 3\vec{e}_1 \wedge \vec{e}_2 - 3\vec{e}_1 \wedge \vec{e}_3 - 3\vec{e}_2 \wedge \vec{e}_3
 \end{aligned}$$

It's often convenient to write a K -vector as

$$\vec{v}_1 \wedge \dots \wedge \vec{v}_K, \text{ where } \vec{v}_i \in V = \Lambda^1(V).$$

Formally, we can compute this by expanding out each $\vec{v}_i = \sum_j (v_i)_j \vec{e}_j$ and using bilinearity of wedge (a lot).

Definition. If V is an inner product space with inner product $\langle, \rangle$, then $\Lambda^K(V)$ is an inner product space with

$$\langle \vec{v}_1 \wedge \dots \wedge \vec{v}_K, \vec{u}_1 \wedge \dots \wedge \vec{u}_K \rangle = \det(\langle \vec{v}_i, \vec{u}_j \rangle).$$

↑
Gram matrix!

Gram Volume

Theorem. If $\vec{v}_1, \dots, \vec{v}_k \in \wedge^k(V)$, then

$$\begin{aligned}\|\vec{v}_1 \wedge \dots \wedge \vec{v}_k\|^2 &:= \langle \vec{v}_1 \wedge \dots \wedge \vec{v}_k, \vec{v}_1 \wedge \dots \wedge \vec{v}_k \rangle \\ &= \text{vol}(\vec{v}_1, \dots, \vec{v}_k)^2\end{aligned}$$

where $\text{vol}(\vec{v}_1, \dots, \vec{v}_k)$ is the signed volume of the parallelipiped spanned by $\vec{v}_1, \dots, \vec{v}_k \in (V, \langle, \rangle)$.

Example.

$$\left\langle \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \wedge \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \wedge \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\rangle = \begin{vmatrix} 6 & 3 \\ 3 & 6 \end{vmatrix} = 27$$

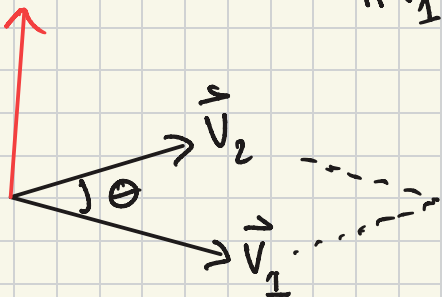
Check.

$$\|\vec{v}_1\|^2 \|\vec{v}_2\|^2 - \langle \vec{v}_1, \vec{v}_2 \rangle^2 =$$

$$= \|\vec{v}_1\|^2 \|\vec{v}_2\|^2 (1 - \cos^2 \theta)$$

$$= \|\vec{v}_1\|^2 \|\vec{v}_2\|^2 \sin^2 \theta$$

$$= \text{area}^2.$$



We can use the inner product to define a special n -vector.

$$\text{vol} = \frac{\vec{v}_1 \wedge \dots \wedge \vec{v}_n}{\|\vec{v}_1 \wedge \dots \wedge \vec{v}_n\|}, \quad \vec{v}_1, \dots, \vec{v}_n \text{ our basis for } V$$

If the inner product on V is defined by the quadratic form Q , this is

$$\text{vol} = \frac{\vec{v}_1 \wedge \dots \wedge \vec{v}_n}{\sqrt{\det Q}}$$

which is "a unit amount of top dimensional volume".

Example.

$$\text{vol} = \frac{\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3}{1} \text{ in } (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$$

standard dot product 

Definition. If $(V, \langle, \rangle)$ is an n -dimensional inner product space, we define $\star: \Lambda^k \rightarrow \Lambda^{n-k}$ by

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \text{vol}$$

for all α , so $\beta \wedge \star \beta = \|\beta\|^2 \text{vol}$.

This is called the Hodge star.

Examples. If $\langle, \rangle$ is the standard dot product and $V = \mathbb{R}^n$, then

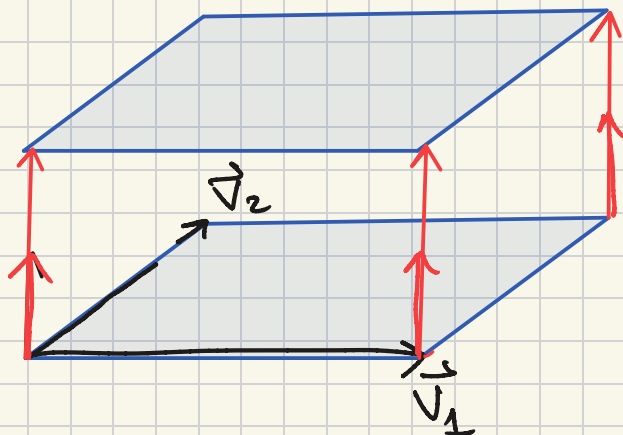
$$\star \vec{e}_{i_1} \wedge \dots \wedge \vec{e}_{i_k} = \vec{e}_{j_1} \wedge \dots \wedge \vec{e}_{j_{n-k}}$$

so that

$$\begin{aligned} (\vec{e}_1 \wedge \dots \wedge \vec{e}_{i_k}) \wedge (\vec{e}_{j_1} \wedge \dots \wedge \vec{e}_{j_{n-k}}) \\ = \vec{e}_1 \wedge \dots \wedge \vec{e}_n = \text{vol}. \end{aligned}$$

Idea:

★ $\vec{v}_1 \wedge \vec{v}_2$
completes the
parallelogram
to a parallelepiped
whose volume is
 $\text{area}(\vec{v}_1, \vec{v}_2)^2$.



Homework: In $\mathbb{R}^3$, ★ $\vec{v}_1 \wedge \vec{v}_2 = \vec{v}_1 \times \vec{v}_2 \in \Lambda^1$.

Definition. The collection of $\Lambda^k(V)$
together with $\wedge$ and ★ form an
exterior algebra on V .

We now consider the dual space.

Definition. If V is a real vector space, then the dual space of V ,

$$V^* := \{ f: V \rightarrow \mathbb{R} \mid f \text{ is linear} \}$$

V^* is a vector space with

$$(f+g)(\vec{v}) := f(\vec{v}) + g(\vec{v})$$

$$(cf)(\vec{v}) := cf(\vec{v})$$

and the elements in V^* are called "covectors".

Example. $V = \text{Mat}_{n \times 1}$, $V^* = \text{Mat}_{1 \times n}$

$$\vec{f}(\vec{v}) = [f_1 \cdots f_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

"row vectors are covectors"

Theorem. If V is a finite-dimensional inner product space, the maps

$$\text{"sharp"} \# : V^* \rightarrow V, \quad \alpha(\vec{v}) = \langle \vec{\alpha}^\#, \vec{v} \rangle$$

$$\text{"flat"} \flat : V \rightarrow V^* \quad w^\flat(\vec{v}) = \langle \vec{w}, \vec{v} \rangle$$

are isomorphisms (called the "musical" isomorphisms).

In tensor notation, vectors have upper indices $\vec{v} = (v^1, \dots, v^n)$ and covectors have lower indices, $\alpha = (\alpha_1, \dots, \alpha_n)$ and the musical isomorphisms raise/lower.

$$\vec{v} = v^i \vec{e}_i, \quad \alpha = \alpha_i dx^i$$

The theorem (and $\#, \flat$) apply to Hilbert spaces (Riesz-Frechet theorem), so

$$V = L^2(S^1), \quad V^* \cong L^2(S^1)$$

In general, things are more complicated

$$V = C^0(S^1) \quad V^* \approx \text{Radon measures on } S^1$$

by Riesz representation theorem.

For example,

$$f \mapsto \int_{S^1} f(x) dx \leftrightarrow \text{Lebesgue measure } dx$$

$$f \mapsto f(\pi/2) \leftrightarrow \text{"atomic" measure at } \pi/2.$$

not a map in $(L^2)^*$ because
 L^2 functions don't have pointwise values.

We now define k -forms to be "multilinear, alternating functionals on K -vectors".

Definition. $(\bigwedge^k(V))^*$ is the $\binom{n}{k}$ dimensional vector space of maps:

$$\alpha: \underbrace{V \times \dots \times V}_{k \text{ copies}} \rightarrow \mathbb{R} \quad \text{or} \quad \alpha: \bigwedge^k(V) \rightarrow \mathbb{R}$$

which are alternating and multilinear.

We write the basis elements for $(\bigwedge^1(V))^*$ as $dx^1, \dots, dx^n$ where

$$dx^i(e_j) = \delta_{ij}$$

and let $\wedge$ be the unique bilinear map $\bigwedge^k(V)^* \times \bigwedge^l(V)^* \rightarrow \bigwedge^{k+l}(V)^*$ defined by

$$\begin{aligned} & (dx^{i_1} \wedge \dots \wedge dx^{i_k}) \wedge (dx^{i_{k+1}} \wedge \dots \wedge dx^{i_{k+l}}) \\ &= \text{sgn}(\sigma) dx^{i_{\sigma(1)}} \wedge \dots \wedge dx^{i_{\sigma(k+l)}} \end{aligned}$$

(or 0 if any index is repeated)

where (as before) σ is the unique permutation that puts these indices in ascending order.

Fact. $\wedge$ is associative
and $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

Useful formula: Given $dx^{i_1} \wedge \dots \wedge dx^{i_k} = \omega$
we can evaluate $\omega(v_1, \dots, v_k)$ by
taking the determinant of the $k \times k$

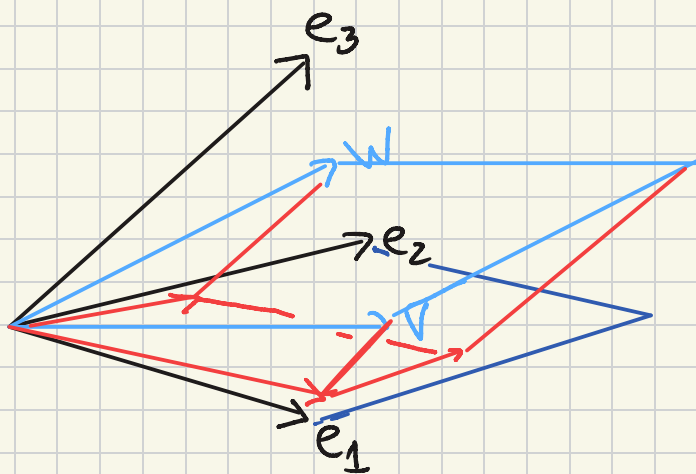
det

$$\begin{bmatrix} \uparrow & \uparrow \\ \text{---} & \text{---} \\ \downarrow & \downarrow \end{bmatrix} \begin{matrix} \leftarrow i_1 \\ \leftarrow i_2 \\ \leftarrow i_k \end{matrix}$$

$\vec{v}_1 \quad \vec{v}_k$

matrix obtained
by selecting
coords $i_1, \dots, i_k$
of the $\vec{v}_j$.

But what does it mean to evaluate $(dx^1 \wedge dx^2)(\vec{v}, \vec{w})$ if V is 3-dimensional?



Geometrically, this amounts to "project v and w along the $\vec{e}_3$ direction to the $\vec{e}_1, \vec{e}_2$ plane and compute signed area of the projection as a fraction of the area of the $\vec{e}_1, \vec{e}_2$ parallelogram".

Fact. If $\alpha_1, \dots, \alpha_k$ are 1-forms and $\vec{v}_1, \dots, \vec{v}_k$ are vectors, then

$$(\alpha_1 \wedge \dots \wedge \alpha_k)(v_1, \dots, v_k) = \det [\alpha_i(v_j)]$$

Note: at no point have we used the inner product on V . That's about to change.

We already know one duality which is independent of $\langle, \rangle$.

$$\underbrace{\bigwedge^1(V)}_{\text{one-forms}} \cong \underbrace{\bigwedge^1(V)}_{\text{one-vectors}}$$

$$dx_i \longleftrightarrow e_i$$

Here's another:

$$\underbrace{\Lambda_1(V)}_{\text{one-forms}} \cong \underbrace{\Lambda^1(V)}_{\text{one-vectors}}$$

$$e_i^b \longleftrightarrow \vec{e}_i$$

where we recall that $e_i^b(v) = \langle \vec{v}, \vec{e}_i \rangle$
and that $\Lambda_1(V) \cong V^*$ as these
are clearly linear functionals (and
alternating doesn't mean anything
when there's only one argument).

Definition. If $(V, \langle, \rangle)$ is a real n -dimensional inner product space and $(\bigwedge^K(V), \langle, \rangle)$ has the induced inner product on K -vectors, the maps

$$\# : \overset{\text{forms}}{\bigwedge^K(V)^*} \rightarrow \overset{\text{vectors}}{\bigwedge^K(V)} \quad (\text{sharp})$$

defined by

$$\alpha(\vec{v}_1 \wedge \dots \wedge \vec{v}_K) = \langle \vec{\alpha}^\#, \vec{v}_1 \wedge \dots \wedge \vec{v}_K \rangle$$

and

$$b : \overset{\text{vectors}}{\bigwedge^K(V)} \rightarrow \overset{\text{forms}}{\bigwedge^K(V)^*} \quad (\text{flat})$$

defined by

$$(\vec{w}_1 \wedge \dots \wedge \vec{w}_K)^b(\vec{v}_1, \dots, \vec{v}_K) = \langle \vec{w}_1 \wedge \dots \wedge \vec{w}_K, \vec{v}_1 \wedge \dots \wedge \vec{v}_K \rangle$$

are vector-space isomorphisms.

Lemma. Suppose $(\alpha^1 \wedge \dots \wedge \alpha^k) \in \bigwedge^k(V)^*$ where the α^i are 1-forms. Then

$$(\alpha^1 \wedge \dots \wedge \alpha^k)^b = \alpha^{1b} \wedge \dots \wedge \alpha^{kb}$$

Proof. Homework.

Definition. If $(V, \langle, \rangle)$ is an n -dimensional real inner product space, then the volume form

$$dVol = \sqrt{\det[g_{ij}, \langle \underline{\tilde{e}}_i, \underline{\tilde{e}}_j \rangle]} dx^1 \wedge \dots \wedge dx^n$$

Lemma. $(dVol)^b = vol$ and $vol^\# = dVol$.

Proof. Homework.

Definition. The inner product on

$\Lambda^K(V)^*$ is defined by

$\uparrow$
K-forms

$$\langle \omega, \eta \rangle := \langle \omega^\#, \eta^\# \rangle$$

where the right-hand side is the inner product on $\Lambda^K(V)$.

$\nwarrow$ K-vectors.

We now have natural bases

vectors

$$e_1, \dots, e_n$$

$\longleftrightarrow$

covectors

$$e_1^b, \dots, e_n^b$$

$$\langle e_i, e_j \rangle = g_{ij} = \langle e_i^b, e_j^b \rangle$$

$$e^1, \dots, e^n \longleftrightarrow dx^1, \dots, dx^n$$

$$\langle e^i, e^j \rangle = g^{ij} = \langle dx^i, dx^j \rangle$$

where $g^{ij} = (g^{-1})_{ij}$, the ij -th element of the inverse matrix.

Proposition. $dx_i(\vec{v}) = v_i = \langle dx_i^\#, \vec{v} \rangle$
and $\langle dx_i, dx_j \rangle := \langle dx_i^\#, dx_j^\# \rangle$
imply $\langle dx_i, dx_j \rangle = g^{ij}$.

Proof. $dx_i(\vec{e}_j) = \delta_{ij} = \langle dx_i^\#, \vec{e}_j \rangle$

Now $(dx_i)^\# = c^{ik} \vec{e}_k$ for some c^{ik} .

So

$$\delta_{ij} = \langle c^{ik} \vec{e}_k, \vec{e}_j \rangle$$

$$= c^{ik} g_{kj}$$

← expanding by linearity, this is

$$\sum_k c^{ik} \langle \vec{e}_k, \vec{e}_j \rangle$$
$$= \sum_k c^{ik} g_{kj}$$

Read as a matrix equation,

$$I_n = cg$$

where $g = [g_{ij}]$ is the Gram matrix of the $\vec{e}_i$ (or the metric) and

$c = [c_{ij}]$ is the matrix of coefficients. But in this form, it's clear that $c = g^{-1}$ as matrices.

Now consider

$$\begin{aligned}\langle dx^i, dx^j \rangle &:= \langle (dx^i)^\#, (dx^j)^\# \rangle \\ &= \langle g^{ik} e_k, g^{jl} e_l \rangle\end{aligned}$$

$$= g^{ik} g^{je} g_{ke} \quad \leftarrow \text{a double sum} \sum_{k,e} g^{ik} g^{je} g_{ke}$$

$$= g^{ik} g_{ke} g^{je} \leftarrow \sum_e \left(\sum_k g^{ik} g_{ke} \right) g^{je}$$

$$= \delta_{ie} g^{je} \leftarrow \sum_e \delta_{ie} g^{je} \quad \begin{array}{l} \text{matrix product} \\ \text{of } g^{-1} \text{ and } g, \\ \text{ie-entry} \end{array}$$

$$= g^{ji} \leftarrow g^{je}$$

$$= g^{ij} \leftarrow (A^{-1})^T = (A^T)^{-1}, \text{ so inverses of symmetric matrices are also symmetric } \square$$

Note: It's reasonable to worry about whether $[g_{ij}]^{-1} = [g^{ij}]$ really defines an inner product. We already showed that it's symmetric.

Since the eigenvalues of $[g_{ij}]^{-1}$ are the reciprocals of the eigenvalues of $[g_{ij}]$, they are positive as well, so $[g_{ij}]^{-1}$ is also positive definite.

Definition. The Hodge star on forms is the unique map $\star: \bigwedge^k(V)^* \rightarrow \bigwedge^{n-k}(V)^*$ defined by

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle d\text{Vol}$$

where $\langle, \rangle$ is the inner product on forms defined above.

Example. (easy)

$$\ast dx^i = \pm \sqrt{\det[g_{ij}]} dx_1 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_n$$

Example (harder).

Let's compute $\ast e_i^b$. We know

$$e_j^b \wedge \ast e_i^b = \langle e_j^b, e_i^b \rangle dVol$$

$$= \langle e_j, e_i \rangle dVol$$

$$= g_{ij} dVol.$$

and we also know that

$$dVol = vol^b$$

$$= \left(\frac{e_1 \wedge \dots \wedge e_n}{\sqrt{\det[g_{ij}]}} \right)^b$$

$$= \frac{1}{\sqrt{\det[g_{ij}]}} e_1^b \wedge \dots \wedge e_n^b$$

which means that

$$\ast(e_i^b) = \sum_j \pm \frac{g_{ij}}{\sqrt{\det[g_{ij}]}} e_1^b \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_n^b$$

Notice that if the $\hat{e}_i$ are orthonormal, this reduces to the previous formula.

We are now ready to transition from "K-vectors and K-forms" to "K-vector **fields** and **differential** K-forms".

You will spare yourself much confusion if you keep repeating "a differential form is a vector-valued map into the vector space of K-forms".

Definition. A Riemannian manifold is a smooth manifold M^n together with a smoothly varying choice of inner product $g(\vec{p})$ on each tangent space $T_p M$.

A differential k-form on the manifold is a smooth map

$$\eta: M \rightarrow \bigwedge^k (T_p M)^*$$

where $T_p M$ has inner product $g(\vec{p})$.

We usually assume that M is parametrized by a map

$$X: U \subset \mathbb{R}^n \rightarrow M$$

so that U is open and $X(\vec{0}) = \vec{p}$.
The coordinates on U are $x^1, \dots, x^n$
and the corresponding basis vectors
for $T_p M$ are $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} = e_1, \dots, e_n$
dual 1-forms $dx^1, \dots, dx^n$.

Each

$$\eta = f^{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where the coefficient functions
are smooth functions

$$f^{i_1, \dots, i_k} : U \rightarrow \mathbb{R}$$

of the local coordinates $x^1, \dots, x^n$.