The First Fundamental Form, lengths, angles, and areas.

We finished last class with a description of a quadratic form as an inner product determined by a symmetric matrix A:

$$Q_A(\vec{v},\vec{w}) = \langle \vec{v}, A\vec{w} \rangle$$

Here is one use of a quadratic form.

Given vectors $\vec{v} = v_1\vec{b}_1 + \dots + v_n\vec{b}_n$
$\vec{w} = w_1\vec{b}_1 + \dots + w_n\vec{b}_n$

written in terms of an arbitrary basis $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ for a subspace $\mathbb{R}^n$ of $\mathbb{R}^K$, how can we compute $\vec{v}\cdot\vec{w}$?

By linearity,

$$\vec{v}\cdot\vec{\omega} = \sum_{i,j=1}^{n} v_i \omega_j (\vec{b}_i \cdot \vec{b}_j)$$

or, if A is the matrix $\begin{bmatrix} \vec{b}_1\cdot\vec{b}_1 & \cdots & \vec{b}_n\cdot\vec{b}_1 \\ \vdots & \ddots & \vdots \\ \vec{b}_1\cdot\vec{b}_n & \cdots & \vec{b}_n\cdot\vec{b}_n \end{bmatrix}$

then

$$\vec{v}\cdot\vec{\omega} = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} A_{ij}\omega_j$$

$$= \langle \vec{v}, A\vec{\omega} \rangle$$

This is the geometric interpretation of the matrix A:

A_{ij} = the inner product of e_i and e_j according to the Q_A inner product

~~On a parame~~

Example. $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. $\vec{v} = (1,2)$, $\vec{w} = (1,1)$.

$$Q_A(\vec{v}, \vec{w}) = (1,2) \cdot A\,(1,1)$$
$$= (1,2) \cdot (3,7) = 17.$$

If $\vec{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrized surface, at any point $(u_0, v_0) = p_0$ on the u-v plane we have the quadratic form determined by

$$A_{p_0} = \begin{bmatrix} \vec{X}_u \cdot \vec{X}_u & \vec{X}_u \cdot \vec{X}_v \\ \vec{X}_v \cdot \vec{X}_u & \vec{X}_v \cdot \vec{X}_v \end{bmatrix}$$

We usually call this matrix

$$A_{p_0} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Definition. The first fundamental form of X at $p = (u_0, v_0)$ is the inner product

$$I_p(\vec{v}, \vec{w}) \quad \text{or} \quad \langle \vec{v}, \vec{w} \rangle_p$$

given by the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$.

Note that $\vec{X}_u, \vec{X}_v$ depend on p, so this matrix changes from point to point.

~~Note as~~

Proposition. I_p is positive-definite.

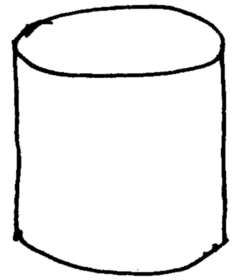
Proof. We know that

$$I_p(\vec{v}, \vec{v}) = (v_1 \vec{X}_u + v_2 \vec{X}_v) \cdot (v_1 \vec{X}_u + v_2 \vec{X}_v)$$

where $\cdot$ is the dot product in $\mathbb{R}^3$ by our discussion above. Hence

$$I_p(\vec{v}, \vec{v}) \geq 0, \text{ with } \vec{v} = 0 \Leftrightarrow I_p(\vec{v}, \vec{v}) = 0.$$

Caution! Do Carmo writes $I_p(\vec{v})$ for $I_p(\vec{v},\vec{v})$.

Example. The cylinder  has the parametrization

$$\vec{X}(u,v) = (\cos u, \sin u, v).$$

We see that

$$\vec{X}_u = (-\sin u, \cos u, 0)$$

$$\vec{X}_v = (0, 0, 1)$$

so

$$E = \vec{X}_u \cdot \vec{X}_u = 1. \quad F = \vec{X}_u \cdot \vec{X}_v = 0.$$

$$G = \vec{X}_v \cdot \vec{X}_v = 1.$$

Example. The xy plane is parametrized by $\vec{X}(u,v) = (u, v, 0)$. We see that

$$\vec{X}_u = (1,0,0) \qquad \vec{X}_v = (0,1,0)$$

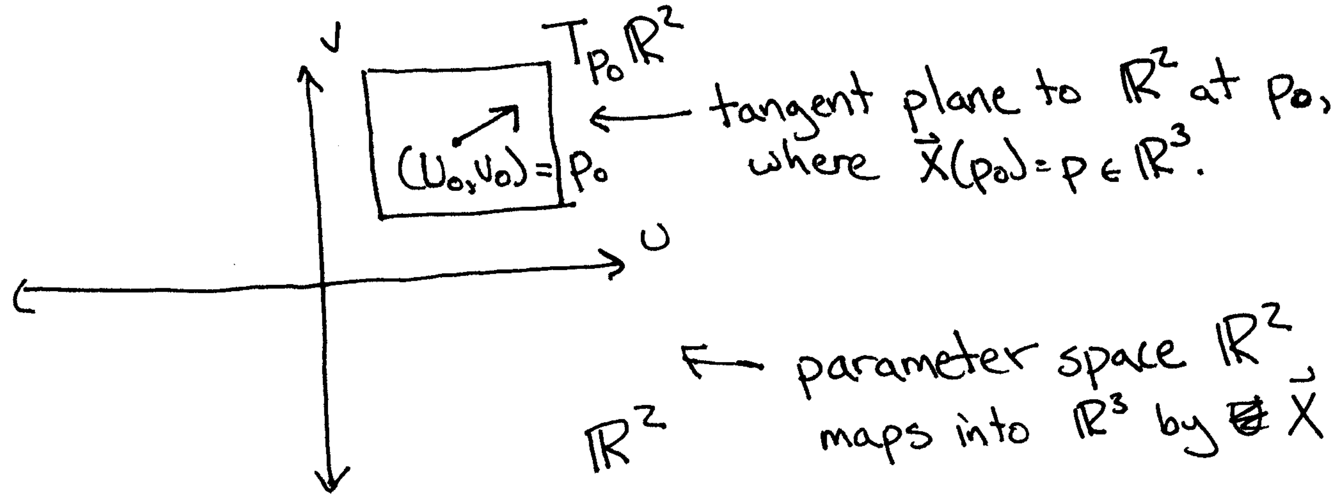
and

$$E = 1 \qquad F = 0 \qquad G = 1.$$

The same result as the cylinder! (This is <u>not</u> a coincidence...)

Computing with I_p

We now make an important but subtle distinction



The tangent plane $T_{p_0}\mathbb{R}^2$ to the u-v plane is a vector space with inner product I_p. We can measure length by I_p, as $\|\vec{v}\|_p = \sqrt{I_p(\vec{v},\vec{v})}$.

The u-v plane does <u>not</u> have a useful inner product. We must measure length by integration.

Let $\alpha(t): \overset{[0,\ell]}{\cancel{I}} \to \mathbb{R}^2$, $\alpha(t) = (u(t), v(t))$ be a parametrized curve on our surface.

Then

$$\text{Length}(\alpha) = \int_0^{\ell} \sqrt{I_p(\alpha'(t),\alpha'(t))}\, dt.$$

$$= \int_0^{\ell} \sqrt{E(u')^2 + 2F\,u'v' + G(v')^2}\, dt$$

This is sometimes written

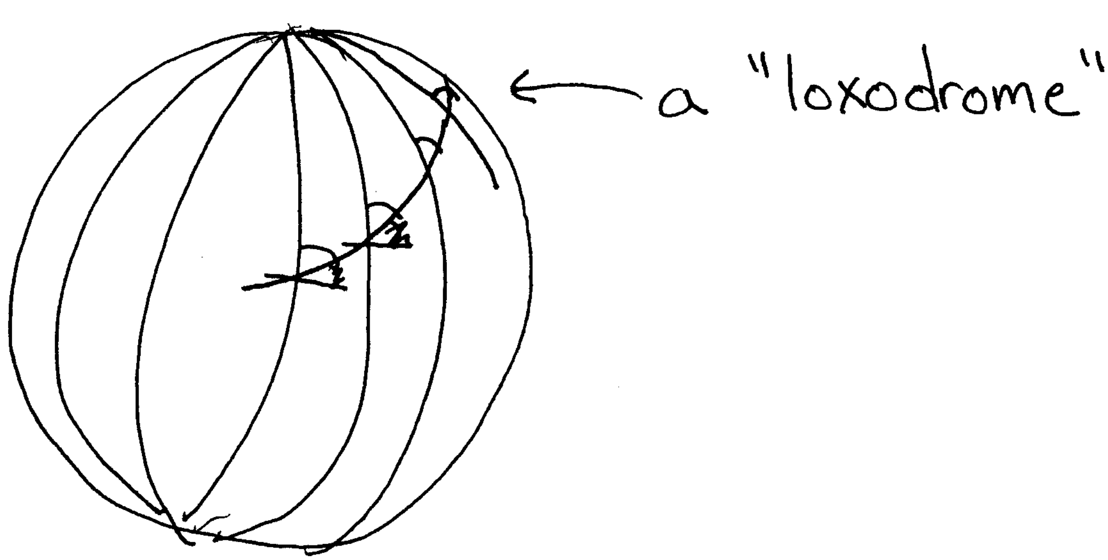
" $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$. "

We can also ~~measure~~ define the angle θ between vectors $\vec{v}$ and $\vec{w}$ in $T_{p_0}\mathbb{R}^2$ by

$$\cos\theta = \frac{I_p(\vec{v},\vec{w})}{\sqrt{I_p(\vec{v},\vec{v})\, I_p(\vec{w},\vec{w})}}$$

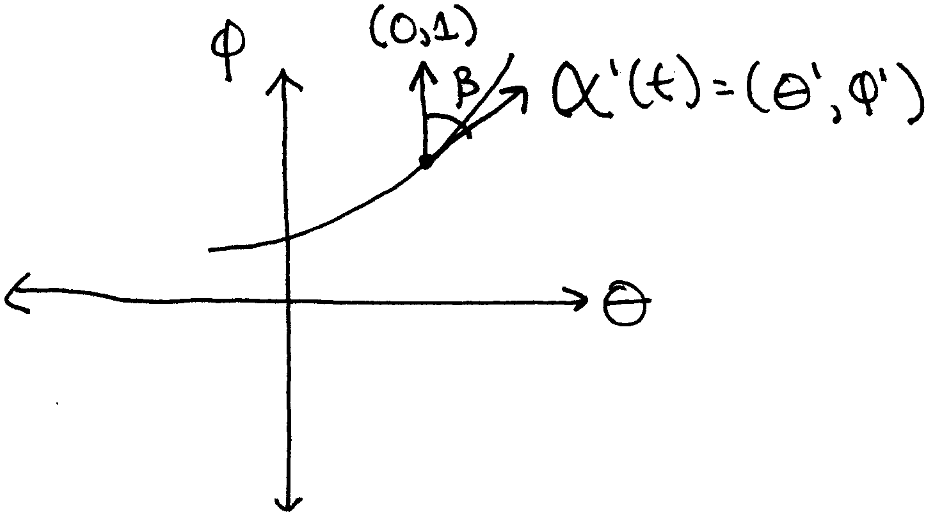
In particular, X_u and X_v are orthogonal $\Leftrightarrow$ $F = I_p\big((1,0),(0,1)\big) = 0$.

Example. Let $X(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ parametrize the sphere. Find the equation of a curve in the (θ,φ) plane which makes a constant angle with the curves $\theta = $ constant.



We compute

$$E = \sin^2\varphi \qquad F = 0 \qquad G = 1$$



Observe that

$$\cos\beta = \frac{I_p((\theta',\varphi'),(0,1))}{\|(0,1)\|_p \, \|\alpha'(t)\|_p}$$

$$= \frac{\varphi'}{1\sqrt{\sin^2\varphi\,(\theta')^2 + (\varphi')^2}}.$$

So we have

$$(\cos^2\beta)(\sin^2\varphi\,(\theta')^2 + (\varphi')^2) = (\varphi')^2$$

$$\cos^2\beta\,(\theta')^2 = \frac{(\varphi')^2(1-\cos^2\beta)}{\sin^2\varphi}$$

or

$$\frac{\theta'}{\tan \cancel{\sin}\, \beta} = \pm \frac{\varphi'}{\sin \varphi}$$

Integrating both sides with respect to t, we get

$$\frac{\theta}{\tan \beta} + c = \pm \ln \tan\left(\frac{\varphi}{2}\right).$$

or

$$\theta = \pm \tan \beta \left(\ln \tan\left(\frac{\varphi}{2}\right)\right) + c$$

where c is determined by the starting point. The integration comes from the half-angle formula

$$\sin \varphi = \sin 2\left(\frac{\varphi}{2}\right) = 2 \sin \varphi/2 \cos \varphi/2$$

so

$$\int \frac{\varphi'(t)\, dt}{2 \sin \varphi/2 \cos \varphi/2} = \int \frac{1}{\sin x \cos x}\, dx = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}\, dx$$

$x = \varphi/2$

$$= \int \frac{1}{\tan x} \cdot \frac{1}{\cos^2 x} \, dx$$

$$= \ln(\tan x) = \ln \tan \varphi/2 .$$

We last consider area on surfaces. In $\mathbb{R}^3$, the area spanned by $\vec{v}$ and $\vec{w}$ is given by $|\vec{v} \times \vec{w}|$. We use this to define

Definition. If $R \subset U$ is a ~~closed~~ bounded region in the parameter plane of a surface given by $\vec{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, then

$$\text{Area}(R) = \iint_R |\vec{X}_u \times \vec{X}_v| \, du\, dv$$

We can use the handy identity

$$|\vec{X}_u \times \vec{X}_v|^2 + (\vec{X}_u \cdot \vec{X}_v)^2 = |\vec{X}_u|^2 \, |\vec{X}_v|^2$$

to write

$$\text{Area}(R) = \iint_R \sqrt{EG - F^2}\, du\, dv$$

where we call $\sqrt{EG-F^2}$ the "element of area".
(This quantity is the determinant of the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ which maps $T_{p_0}\mathbb{R}^2 \to T_p S$.)

Example. The area of a sphere.