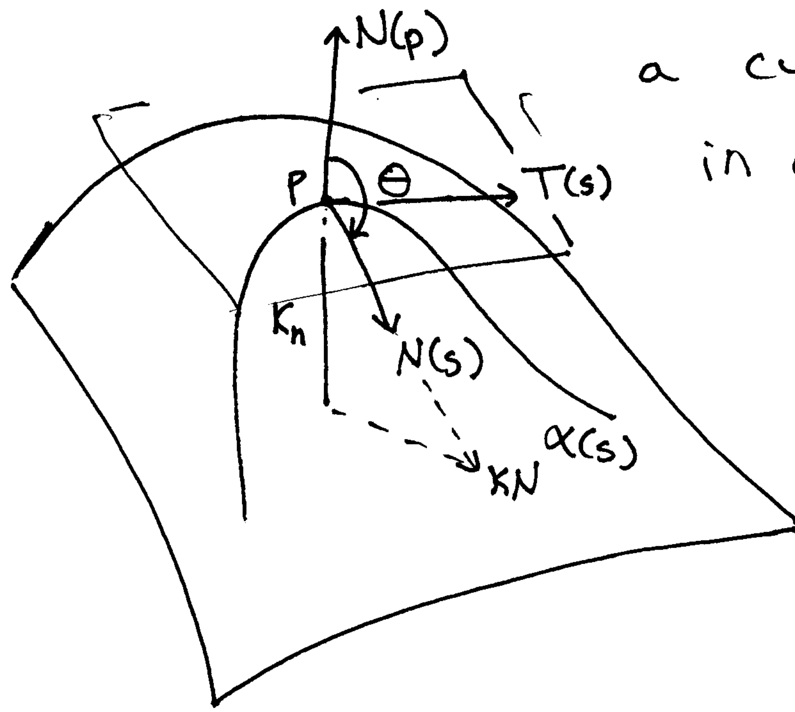
The meaning of the second fundamental form - what is this guy telling us?

We recall that

$$\mathrm{II}_p(\vec{v},\vec{w}) = \langle -dN_p(\vec{v}), \vec{w} \rangle_{I_p}$$

is the second fundamental form of our surface.

What does it mean? Suppose we have a curve $\alpha(s)$ in our surface S.



Then $T(s) = \alpha'(s)$ is contained in T_pS. The normal vector $N(s)$ to $\alpha(s)$ does

<u>not</u> have to lie in the tangent plane surface.
In fact, it usually doesn't.

Definition. The <u>normal</u> <u>curvature</u> of $\alpha(s)$ at s_0 is given by

$$K_n(s_0) = \langle \underbrace{K(s_0)N(s_0)}_{\text{curve normal}}, \underbrace{N_{\alpha(s_0)}}_{\text{surface normal}} \rangle$$

where

$N(s_0)$ is the normal vector to the <u>curve</u>, and $N_{\alpha(s_0)}$ is the normal to the <u>surface</u>.

Proposition. $K_n(s_0) = \mathrm{II}_{\alpha(s_0)}(\alpha'(s_0))$.

Proof. We see that

$$K_n(s_0) = \langle \alpha''(s_0), N_{\alpha(s_0)} \rangle$$

But $\langle \alpha'(s), N_{\alpha(s)} \rangle \equiv 0$, so

$$0 = \frac{d}{ds} \langle \alpha'(s), N_{\alpha(s)} \rangle = \langle \alpha''(s), N_{\alpha(s)} \rangle + \langle \alpha'(s), N'_{\alpha(s)} \rangle$$

So

$$K_n(s_0) = -\left\langle \alpha'(s_0), N'_{\alpha(s)}\Big|_{s=s_0} \right\rangle.$$

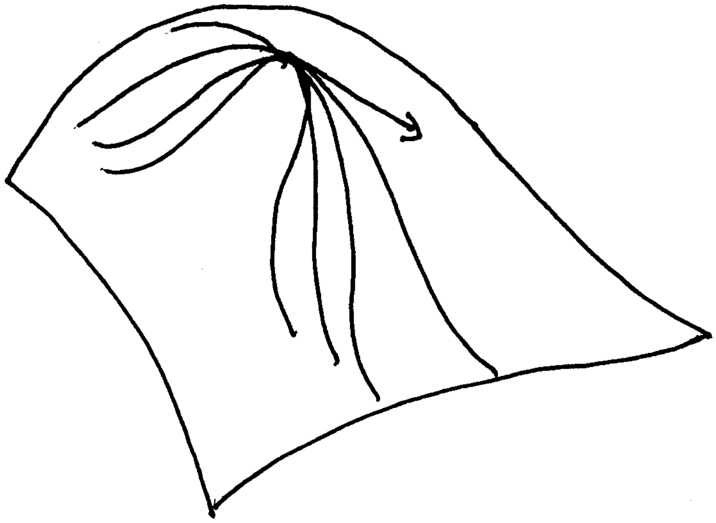
But $N'_{\alpha(s)}$ is the change in N as we move in the direction $\alpha'(s)$. This is the <u>definition</u> of $dN_{\alpha(s)}(\alpha'(s))$.

So

$$K_n(s_0) = -\langle \alpha'(s_0), dN_{\alpha(s_0)}(\alpha'(s_0)) \rangle$$
$$= \mathrm{II}_p(\alpha'(s_0)).$$

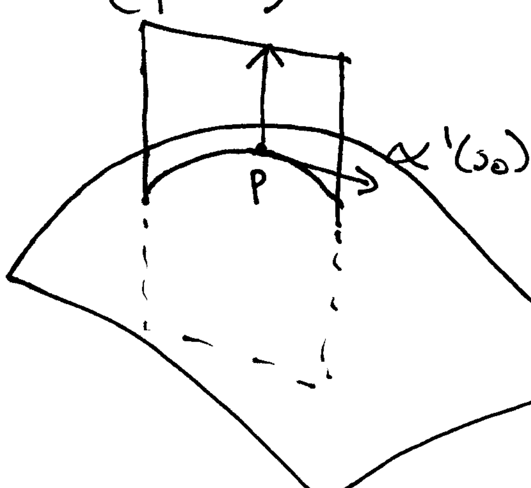
Notice the cool trick of switching a derivative from α'' to N allowed us to compute a <u>second</u> derivative (K_n) in terms of first derivatives (α', dN_p).

This fact tells us something cool about curves in a surface.



Proposition (Meusnier's theorem) All curves in S through p with the same tangent vector at p have the same normal curvature at p.

In fact, one of these curves has the <u>least</u> possible (space) curvature:



The intersection of a plane containing n_p and $\alpha'(s_0)$ with S is called the <u>normal</u> <u>section</u> of S in direction α!

The curvature of the normal section

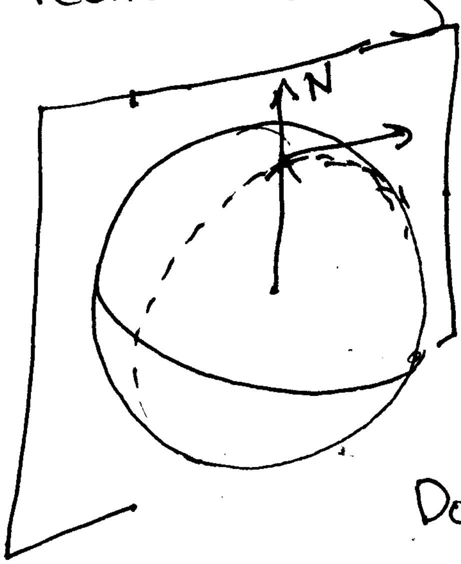
$\parallel$

the normal curvature of any curve in S with the same tangent line

$\parallel$

$$\mathrm{II}_p(\alpha'(s_0)), \quad \left(\text{assuming } \alpha' \text{ is a } \underline{\text{unit}} \text{ vector}\right)$$

Examples. We can compute II_p without tedium using normal sections. For S^2,



the normal sections are unit circles, so

$$\mathrm{II}_p(\vec{v}) = 1$$

for all $\vec{v}$ with norm 1.

Does that determine II_p?

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Trick. For any quadratic form Q, we have

$$Q(\vec{v},\vec{\omega}) = \frac{1}{2}\left[Q(\vec{v}+\vec{\omega}) - Q(\vec{v}) - Q(\vec{\omega})\right]$$

Proof of trick.

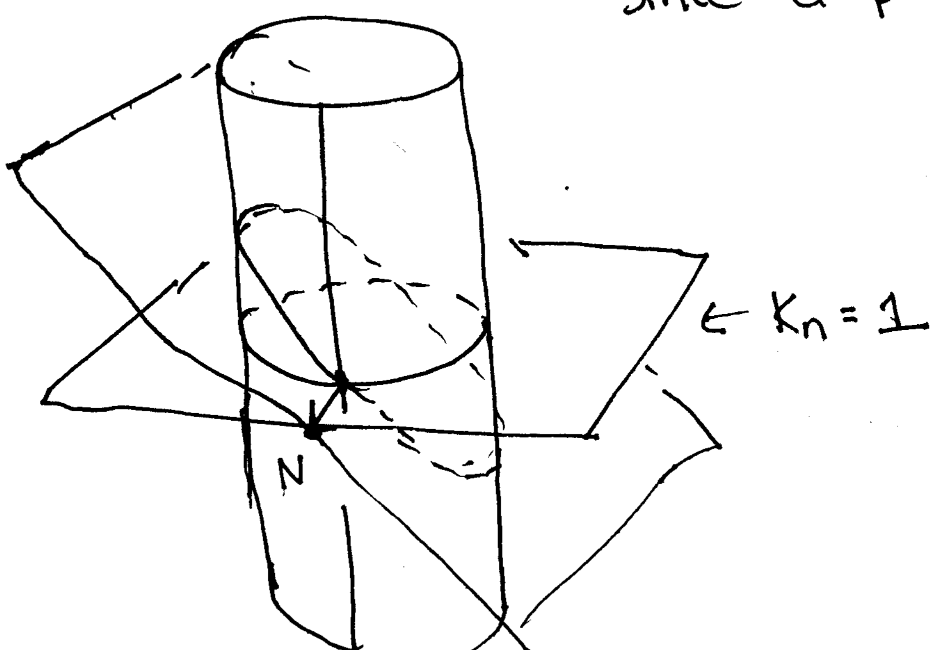
$$Q(\vec{v}+\vec{\omega}, \vec{v}+\vec{\omega}) = Q(\vec{v},\vec{v}) + 2Q(\vec{v},\vec{\omega}) + Q(\vec{\omega},\vec{\omega})$$

so we can solve for $Q(\vec{v},\vec{\omega})$. $\therefore$

Thus since

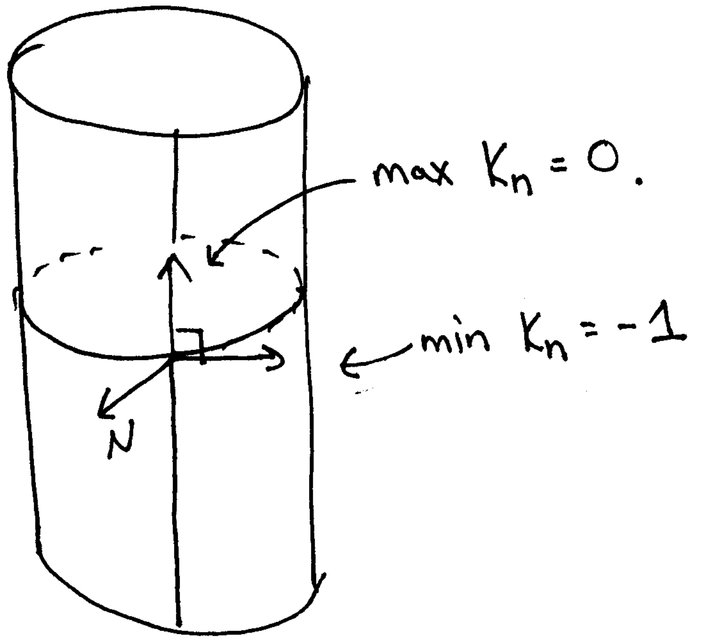
$$\mathrm{II}_p(\vec{v}) = -\mathrm{I}_p(\vec{v})$$

for all $\vec{v}$, we have $\mathrm{II}_p(\vec{v},\vec{\omega}) = -\mathrm{I}_p(\vec{v},\vec{\omega})$ for all $\vec{v},\vec{\omega}$ and $-\mathrm{II}_p$ is the identity matrix, since dN_p is the identity.



For the cylinder, K_n varies from -1 to 0, depending on which direction is chosen in T_pS.

We notice that the directions of maximum and minimum normal curvature are orthogonal.



This is an instance of a general theorem.

Proposition. Given two quadratic forms Q and P on $\mathbb{R}^2$, there is a basis $\vec{v}_1, \vec{v}_2$ of $\mathbb{R}^2$ so that

$$\langle \vec{v}_i, \vec{v}_j \rangle_P = \delta_{ij} \quad \text{or} \quad P(v_i, v_j) = \delta_{ij}$$

(the basis is orthonormal w.r.t. P) and

we have

$$Q(v_1) = \lambda_1$$
$$Q(v_2) = \lambda_2$$

are the max and min values of Q on the "unit circle"

$$\vec{\omega} = \cos\theta \, \vec{v}_1 + \sin\theta \, \vec{v}_2 .$$

Proof. Suppose that we have a quadratic function $Q(x,y) = ax^2 + 2bxy + cy^2$. If this function has a max on $x^2+y^2=1$ at $(1,0)$, then $b=0$.

Parametrize $x^2+y^2=1$ by $(\cos\theta, \sin\theta)$. We have

$$\frac{d}{d\theta} Q(\cos\theta, \sin\theta) = 2a \cos\theta(-\sin\theta)$$
$$+ 2b(-\sin^2\theta + \cos^2\theta)$$
$$+ 2c \sin\theta \cos\theta .$$

Evaluating at $\theta = 0$, we get

$$= 2b.$$

So if $\theta = 0$ (or $(1,0)$) is a max, then $b = 0$.

We now prove our main result.

Suppose we define the unit circle of vectors $\vec{v}_1$ with $P(\vec{v}) = 1$ and find the max of $Q(\vec{v})$ on this set. This is at some $\vec{v}_1$.

Choose $\vec{v}_2$ so that $P(\vec{v}_1, \vec{v}_2) = 0$, $P(\vec{v}_2) = 1$.

We claim that

$$Q(x\vec{v}_1 + y\vec{v}_2) = x^2 Q(\vec{v}_1) + y^2 Q(\vec{v}_2).$$

We know

$$Q(x\vec{v}_1 + y\vec{v}_2) = x^2 Q(\vec{v}_1) + 2xy\, Q(\vec{v}_1, \vec{v}_2) + y^2 Q(v_2).$$

$$= x^2 a + 2xyb + y^2 c = Q(x,y)$$

Since $(1,0)$ is the max of $Q(x,y)$, $b = 0$.

Thus

$$Q(x\vec{v}_1 + y\vec{v}_2) = x^2 Q(\vec{v}_1) + y^2 Q(\vec{v}_2).$$

We need only show that

$$Q(\vec{v}_2) \text{ is the min of } Q(\vec{v}) \text{ on } P(\vec{v}) \equiv 1.$$

This is easy: $Q(\vec{v}_2) \leq Q(\vec{v}_1)$, since $Q(\vec{v}_1)$ was chosen to be the <u>max</u> of $Q(\vec{v})$ on $P(\vec{v}) = 1$.

So

$$\begin{aligned} Q(\cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2) &= \cos^2\theta\, Q(\vec{v}_1) + \sin^2\theta\, Q(\vec{v}_2) \\ &\geq Q(\vec{v}_2)\left(\cos^2\theta + \sin^2\theta\right) \\ &\geq Q(\vec{v}_2). \end{aligned}$$

Though we won't prove it,

$$\vec{v}_1 \text{ and } \vec{v}_2 \text{ are eigenvectors for } dN_p.$$

So in this basis, if

$$\begin{aligned} dN_p(\vec{v}_1) &= -k_1 \vec{v}_1 \\ dN_p(\vec{v}_2) &= -k_2 \vec{v}_2 \end{aligned}, \quad \text{then} \quad \begin{aligned} \mathrm{II}_p(\vec{v}_1) &= k_1 \\ \mathrm{II}_p(\vec{v}_2) &= k_2 \end{aligned}$$

are the maximum and minimum normal

curvatures at p.

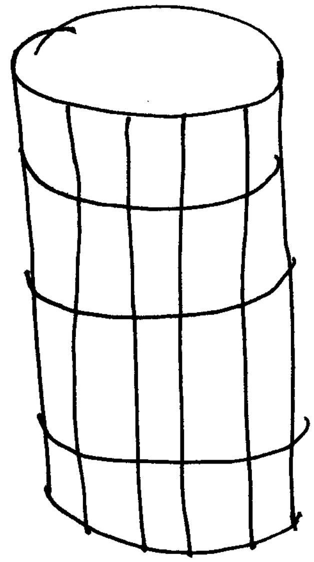
<u>Definition.</u> The maximum (k_1) and minimum (k_2) normal curvatures are called the <u>principal curvatures</u> at p.

The directions $\vec{v}_1$ and $\vec{v}_2$ are called <u>principal directions</u>.

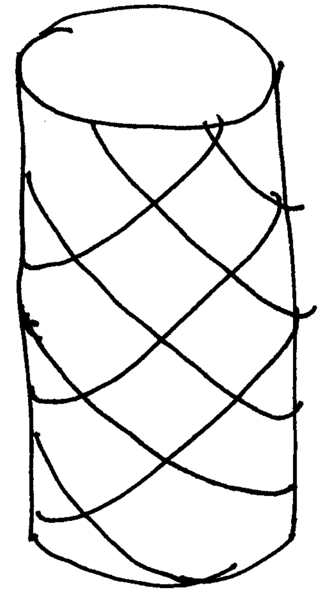
We can define some special coordinate lines on our surface by

<u>Definition</u>. If $\alpha(s) \in S$ is a curve on S so that $\alpha'(s)$ is in a principal direction for S at $\alpha(s)$, then α is a <u>line of curvature</u>.

Example.



lines of curvature



not lines of curvature.

Proposition. $\alpha(s)$ is a line of curvature $\Leftrightarrow$ $N'(t) = \lambda(t)\,\alpha'(t)$, for some differentiable function $\lambda(t)$. $-\lambda(t)$ is the curvature along $\alpha(t)$.

Proof. If $\alpha(t)$ is a line of curvature,

$$N'(t) = dN(\alpha'(t)) = \lambda(t)\alpha'(t),$$

since $\alpha'(t)$ must be an eigenvector of dN. Further, the eigenvalue $\lambda(t)$ is $-$ the principal curvature in that direction.

We know that elementary symmetric functions of the eigenvalues of a matrix are the coefficients of the characteristic polynomial. Further, these are invariant under change of basis. So we take

Definition. The <u>Gaussian curvature</u> of S is given by

$$K = \kappa_1 \kappa_2 \quad \text{(product of principal curvatures)}$$

and the <u>mean curvature</u> is

$$H = \frac{\kappa_1 + \kappa_2}{2} \quad \text{(average of principal curvatures)}$$

These are the determinant and trace of dN_p as a map from T_pS to T_pS.

<u>Beware</u>! If $N_u = aX_u + cX_v$, $N_v = bX_u + dX_v$ so we write $dN_p = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det dN_p$ is <u>not</u> $ad-bc$.

The problem is that X_u, X_v is not an <u>orthonormal</u> basis for T_pS, so the formula for det is different.