Computations with K and H

We have now learned that

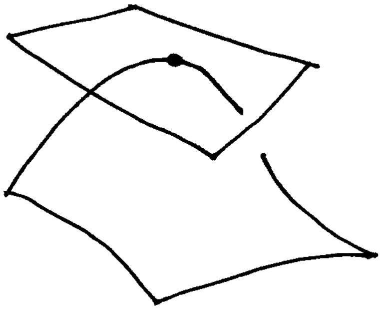
$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{(eG + gE) - 2fF}{2(EG - F^2)},$$

and given nice formulae for the E, F, G and e, f, g:

$$E = \langle x_u, x_u \rangle \qquad e = \langle x_u, N_u \rangle$$
$$F = \langle x_u, x_v \rangle \qquad f = = \langle x_u, N_v \rangle$$
$$G = \langle x_v, x_v \rangle \qquad g = \langle x_v, N_v \rangle$$

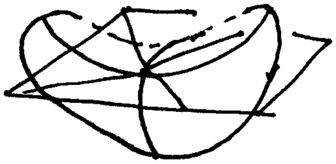
We want to see how to draw some geometric information from these.

Observation.



T_pS lies on one side of S

elliptic



T_pS cuts S

hyperbolic

Let's view II_p in a new light.

When we defined I_p, we proved that the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ and the form I_p were "positive-definite", meaning that

$$\mathrm{I}_p(\vec{v}) \geq 0, \quad \mathrm{I}_p(\vec{v}) = 0 \iff \vec{v} = \vec{0}.$$

We recall

M is positive definite $\iff$ all M's eigenvalues are > 0.

A form is <u>negative definite</u> if

$$Q(\vec{v}) \leq 0, \quad Q(\vec{v}) = 0 \iff \vec{v} = \vec{0}$$

and it is <u>definite</u> if positive or negative definite. A form is <u>indefinite</u> if

$$Q(\vec{v}) < 0, \quad Q(\vec{w}) > 0 \quad \text{for some } \vec{v}, \vec{w}.$$

We can then give a new interpretation for curvature

p is elliptic

$$\Updownarrow$$

$$K(p) > 0 \Leftrightarrow K_1 K_2 > 0$$

$$\Updownarrow$$

II_p is definite

p is hyperbolic

$$\Updownarrow$$

$$K(p) < 0 \Leftrightarrow K_1 K_2 < 0$$

$$\Updownarrow$$

II_p is indefinite

at parabolic ~~a~~ points, II_p turns out to be <u>semi</u> definite, meaning

$$\mathrm{II}_p(\vec{v}) \geq 0 \quad \text{for all } \vec{v} \neq 0$$

or

$$\mathrm{II}_p(\vec{v}) \leq 0 \quad \text{for all } \vec{v} \neq 0$$

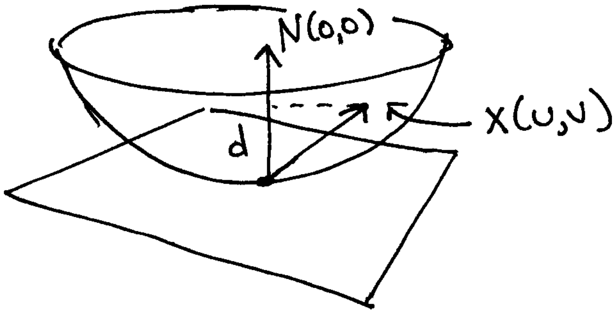
Proposition. If $p \in S$ is an elliptic point, $\exists$ some neighborhood V of p in S so that all points in V belong to the same side of T_pS.

If $p \in S$ is a hyperbolic point, then any neighborhood V of p contains points on both sides of T_pS.

Proof. Suppose that $p = x(0,0) = (0,0,0)$. Then

$$x(u,v) = x_u u + x_v v + \frac{1}{2}\left(x_{uu}u^2 + 2x_{uv}uv + x_{vv}v^2\right) + R(u,v)$$

where $\lim_{(u,v)\to 0} \frac{R(u,v)}{|(u,v)|^2} \to 0$, by Taylor's Theorem.



The height of $x(u,v)$ over the tangent plane is given by

$$\begin{aligned} d &= \langle x, N\rangle \\ &= \langle x_u, N\rangle u + \langle x_v, N\rangle v + \frac{1}{2}\Big(\langle x_{uu}, N\rangle u^2 + 2\langle x_{uv}, N\rangle uv \\ &\qquad + \langle x_{vv}, N\rangle v^2\Big) \\ &\quad + \langle R, N\rangle. \end{aligned}$$

$$= \frac{1}{2}\left(e u^2 + 2f\, uv + g v^2\right) + \langle R, N\rangle$$

$$= \frac{1}{2} II_p\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) + \langle R, N\rangle.$$

If p is an elliptic point, then II_p is positive or negative definite and $\frac{1}{2}II_p(\vec{\omega})$ has the same sign for all $\vec{\omega}$.

If p is a hyperbolic point, then II_p is indefinite and $II_p(\vec{\omega})$ takes different signs for different vectors.

For small enough (u,v), we can ignore $\langle R,N\rangle$ and these observations prove the result.

Let's go back to

asymptotic directions $\Leftrightarrow$ $\mathrm{II}_p(\vec{v}) = 0$.

We see that $\alpha(t) = (u(t), v(t))$ is an asymptotic curve $\Leftrightarrow$

$$e(u')^2 + 2f(u'v') + g(v')^2 \equiv 0.$$

This is a particular differential equation for $\alpha'(t)$.

Example. On the torus of revolution

$$e = r, \quad f = 0, \quad g = \cos u (a + r\cos u)$$

so $\alpha(t)$ is an asymptotic curve $\Leftrightarrow$

$$r(u')^2 + \cos u (a + r\cos u)(v')^2 = 0$$

or

$$\frac{d}{dv} u(v) = \frac{u'}{v'} = -\frac{\cos u}{r}(a + r\cos u)$$

We could solve this differential equation for $u(v)$.

(Can it be solved when $\cos u > 0$?)

What about principal directions?

Recall that $\alpha(t)$ is a line of curvature

$\Leftrightarrow$ $dN_p(\alpha'(t)) = \lambda(t)\,\alpha'(t)$.

or if $\alpha'(t) = (u'(t), v'(t))$, then

$$a_{11}\,u' + a_{12}\,v' = \lambda u' \qquad (1)$$

$$a_{21}\,u' + a_{22}\,v' = \lambda v' \qquad (2)$$

Multiplying (1) by v' and (2) by u', and subtracting them, we get

$$-a_{21}(u')^2 + (a_{12} - a_{22})\,v'u' + a_{12}(v')^2 = 0$$

or (using the Weingarten equations)

$$(fE - eF)(u')^2 + (gE - eG)\,u'v' + (gF - fG)(v')^2 = 0$$

Example. On the torus of revolution,

$$E = r^2 \qquad F = 0 \qquad G = (a + r\cos u)^2$$

so we have

$$\left[r^2 \cos u\,(a + r\cos u) - r(a + r\cos u)^2\right] u'v' = 0$$

or

$$r(a + r\cos u)\left[r \overset{\cos u}{\swarrow} - a \mp r\cos u\right] u'v' = 0$$

or

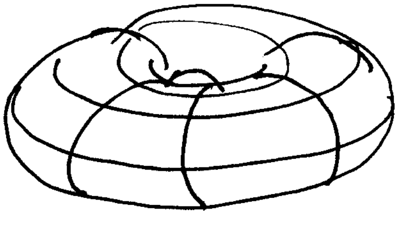
$$-r(a + r\cos u)\left[a + \cancel{r(\cos u - 1)}\right] u'v' = 0$$

~~This tells us that $u' = 0$, $v' = 0$ or~~

~~$a + r(\cos u) = r$~~

⇒ ~~$\cos u = -\frac{a}{r}$, which is impossible, since $a > r$.~~

This tells us that $u' = 0$ or $v' = 0$ and the lines of curvature are



Notice that what we really needed here was the observation that

$$F = f = 0.$$

In fact,

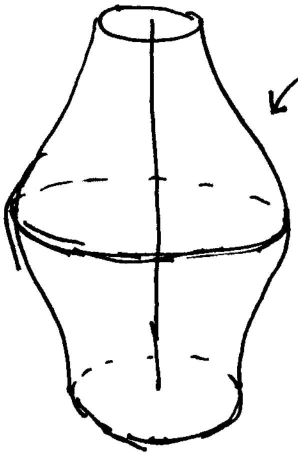
Coordinate curves are lines of curvature

$$\Updownarrow$$

$$F = f = 0$$

Some other special cases are instructive.

Surfaces of Revolution.



$$X(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)).$$

Working out the coefficients E, F, G, e, f, g we compute

$$X_u = (-\varphi(v)\sin u, \varphi(v)\cos u, \cancel{\varphi}\, 0)$$
$$X_v = (\varphi'(v)\cos u, \varphi'(v)\sin u, \psi'(v))$$

$$X_{uu} = (-\varphi(v)\cos u, -\varphi(v)\sin u, 0)$$

$$X_{uv} = (-\varphi'(v)\sin u, \varphi'(v)\cos u, 0)$$

$$X_{vv} = (\varphi''(v)\cos u, \varphi''(v)\sin u, \psi''(v)).$$

So

$$E = \varphi^2(v). \quad F = 0. \quad G = (\varphi')^2 + (\psi')^2 = |\alpha'(v)|^2.$$

We usually assume that $\alpha(v)$ is parametrized by arclength, so $G = 1$. Thus

$$\sqrt{EG-F^2} = \varphi$$

and we have

$$e = \frac{1}{\varphi}\begin{vmatrix} X_u & X_v & X_{uu} \end{vmatrix} = \frac{1}{\varphi}\begin{vmatrix} -\varphi\sin u & \varphi'\cos u & -\varphi\cos u \\ \varphi\cos u & \varphi'\sin u & -\varphi\sin u \\ 0 & \psi' & 0 \end{vmatrix}$$

$$= \frac{1}{\varphi}\left(-\psi'(\varphi^2\sin^2 u - (-\varphi^2\cos^2 u))\right)$$

$$= -\varphi\psi'.$$

$$f = \frac{1}{\varphi}(x_u, x_v, x_{uv}) = \frac{1}{\varphi}\begin{vmatrix} -\varphi \sin u & \varphi' \cos u & -\varphi' \sin u \\ \varphi \cos u & \varphi' \sin u & \varphi' \cos u \\ 0 & \psi' & 0 \end{vmatrix}$$

$$= \frac{1}{\varphi}\left(-\psi'\left(-\varphi\varphi' \sin u \cos u - (-\varphi\varphi' \sin u \cos u)\right)\right)$$

$$= 0.$$

$$g = \frac{1}{\varphi}(x_u, x_v, x_{vv}) = \frac{1}{\varphi}\begin{vmatrix} -\varphi \sin u & \varphi' \cos u & \varphi'' \cos u \\ \varphi \cos u & \varphi' \sin u & \varphi'' \sin u \\ 0 & \psi' & \psi'' \end{vmatrix}$$

$$= \frac{1}{\varphi}\left(-\psi'(-\varphi\varphi'' \sin^2 u - \varphi\varphi'' \cos^2 u) + \psi''(-\varphi\varphi' \sin^2 u - \varphi\varphi' \cos^2 u)\right)$$

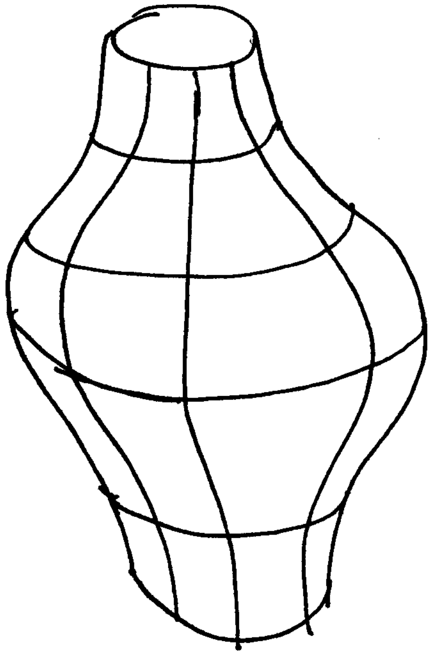
$$= \psi'\varphi'' - \psi''\varphi'.$$

Notice that g should look familiar! From your old homework, the curvature of $\alpha(v) = (\varphi(v), \psi(v))$

$$K(v) = \frac{|\varphi'\psi'' - \varphi''\psi'|}{((\varphi')^2 + (\psi')^2)^{3/2}} \underset{\uparrow}{=} |\varphi'\psi'' - \varphi''\psi'|$$

since (φ, ψ) is unit speed.

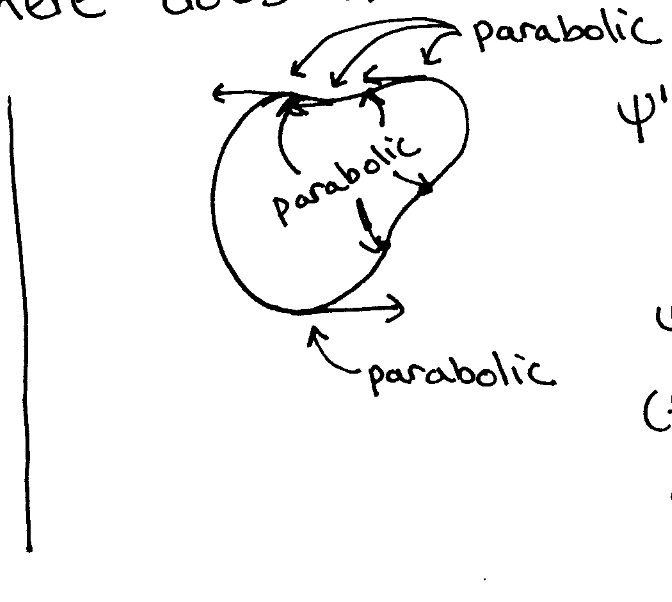
We know that the coordinate curves are lines of curvature here, too:



$$K = \frac{eg - f^2}{EG - F^2} = \frac{-\varphi\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi^2}$$

$$= -\frac{1}{\varphi}\,\psi'(\psi'\varphi'' - \psi''\varphi')$$

So where does K vanish? (Parabolic points?)



$\psi' = 0$

or

$\psi'\varphi'' - \psi''\varphi' = 0$

(the curve α has a point of zero curvature)

We can come up with a better expression for K. Notice that

$$\varphi'^2 + \psi'^2 \equiv 1,$$

so differentiating w.r.t. v, we get

$$2\varphi'\varphi'' + 2\psi'\psi'' = 0, \text{ or } \varphi'\varphi'' = -\psi'\psi''$$

Thus

$$K = \frac{-(\psi')^2\varphi'' + \psi'\psi''\varphi'}{\varphi} = \frac{(-(\psi')^2 + -(\varphi')^2)\varphi''}{\varphi} = -\frac{\varphi''}{\varphi}.$$

Application.

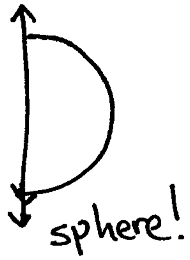
We then know that a surface of revolution has constant Gaussian curvature K_0 if (and only if)

$$-\frac{\varphi''}{\varphi} = K_0, \quad \text{or} \quad \varphi'' = -K_0\varphi$$

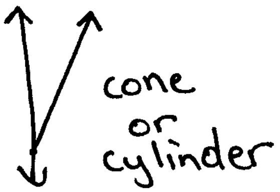
What are the solutions of this O.D.E.?

$K_0 = +1 \qquad \varphi'' = -\varphi. \qquad \varphi(v) = \sin v$ or $\cos v$

$\psi(v) = \cos v$ or $\sin v$

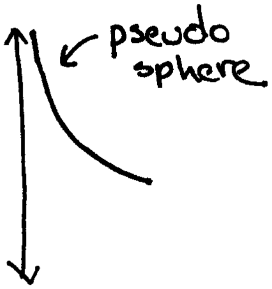


$K_0 = 0 \qquad \varphi'' = 0. \quad \varphi(v) = av + b.$



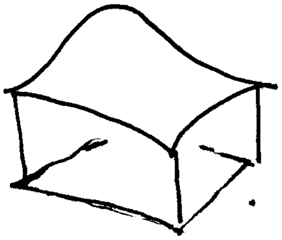
$K_0 = -1 \quad \varphi'' = \varphi. \qquad \varphi(v) = \cosh v$ or $\sinh v$

$$\psi(v) = \int \sqrt{1 - \sinh^2 v}\, dv$$



It will be a homework exercise to work out a better parametrization for this surface!

Example. Surfaces which are graphs of functions $h(u,v)$.



$$X(u,v) = (u, v, h(u,v))$$

so

$$X_u = (1, 0, h_u)$$
$$X_v = (0, 1, h_v)$$

$$X_{uu} = (0, 0, h_{uu})$$
$$X_{uv} = (0, 0, h_{uv})$$
$$X_{vv} = (0, 0, h_{vv})$$

Here

$$E = 1 + (h_u)^2 \quad F = h_u h_v \quad G = 1 + (h_v)^2$$

then

$$EG - F^2 = (1 + (h_u)^2)(1 + (h_v)^2) - (h_u^2)(h_v)^2$$
$$= 1 + (h_u)^2 + (h_v)^2.$$

so

$$(x_u, x_v, x_{uu}) = \begin{vmatrix} 1 & \cancel{1}0 & 0 \\ 0 & 1 & 0 \\ h_u & h_v & h_{uu} \end{vmatrix} = h_{uu}$$

$$(x_u, x_v, x_{uv}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h_u & h_v & h_{uv} \end{vmatrix} = h_{uv}$$

$$(x_u, x_v, x_{uv}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h_u & h_v & h_{vv} \end{vmatrix} = h_{vv}$$

and

$$e = \frac{h_{uu}}{(1 + h_u^2 + h_v^2)^{1/2}}$$

$$f = \frac{h_{uv}}{(1 + h_u^2 + h_v^2)^{1/2}}$$

$$g = \frac{h_{vv}}{(1 + h_u^2 + h_v^2)^{1/2}}$$

This means that

$$K = \frac{eg-f^2}{EG-F^2} = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1+h_u^2+h_v^2)^2}.$$

In particular, we see that the sign of K is given by the determinant of the Hessian matrix

$$H = \begin{bmatrix} h_{uu} & h_{uv} \\ h_{vu} & h_{vv} \end{bmatrix},$$

telling us that the Hessian and the second fundamental form are related! In fact given any direction $\vec{v}$, the second derivative of h in that direction is given by

$$\langle H\vec{v}, \vec{v} \rangle = \frac{d^2}{d\epsilon^2} h(\vec{x}+\epsilon\vec{v}) \Big|_{\epsilon=0}$$