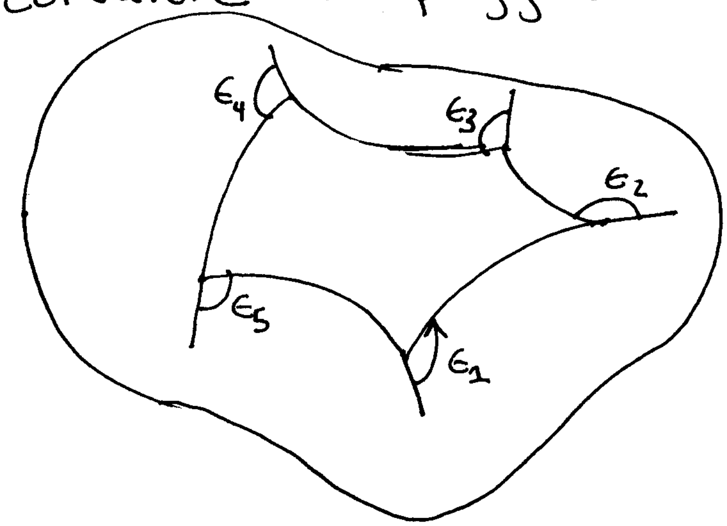
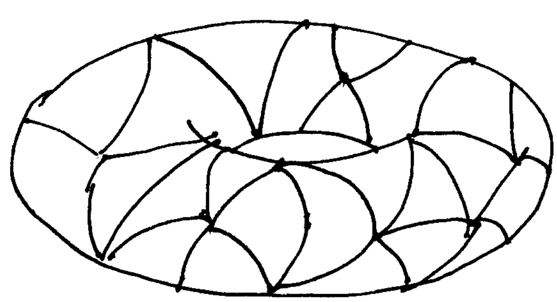
Putting the pieces together: Global Differential Geometry (of Surfaces)

The Gauss-Bonnet theorem related geodesic curvature, exterior angles, and total Gauss curvature for polygons on a surface:



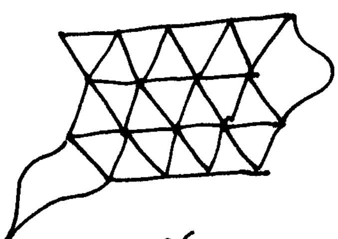
$$\sum \int_{\alpha_i} \kappa_g(s)\,ds + \iint_R K\,dA = 2\pi - \sum_i \epsilon_i$$

We now want to tile the entire surface with these polygons and see what we get.

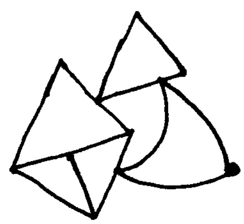


Definition. A surface S has a <u>triangulation</u> $P = \{\Delta_i \subset S\}$ if each Δ_i is the image of a triangle in the u-v plane, and $\Delta_i \cap \Delta_j$ is either empty, a shared vertex, or a shared side.

Examples.

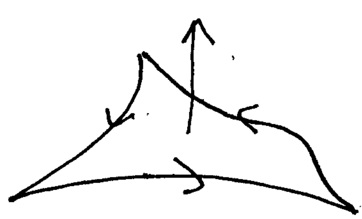


yes



no

We orient the boundary of each triangle so that the interior is to the left.



Given a triangulation, let

F = # of triangles (faces)

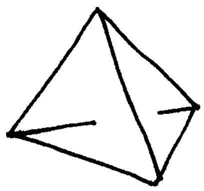
E = # of sides (edges)

V = # of vertices

We define

Definition. The <u>Euler characteristic</u> $\chi_p(S)$ of a triangulated surface S is $V - E + F$.

Now Euler characteristic is an amazing story on its' own. We don't have time to really tell this tale, so we do a few examples.



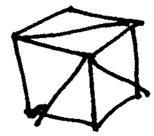
tetrahedron

$V = 4$

$E = 6$

$F = 4$

$\chi = 2$



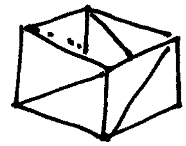
cube

$V = 8$

$E = 18$

$F = 12$

$\chi = 2$



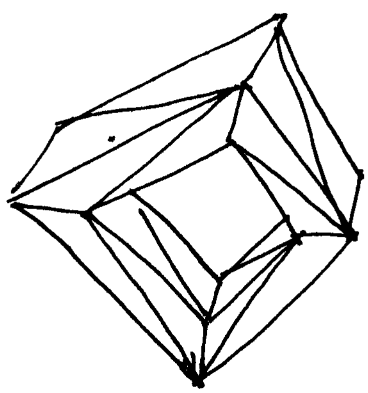
~~cube~~ open cube

$V = 8$

$E = 16$

$F = 8$

$\chi = 0$



picture frame

$V = 16$

$E = 48$

$F = 32$

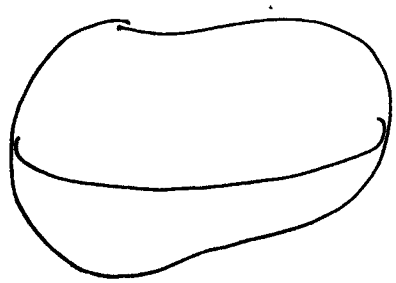
$\chi = 0$

An amazing fact:

1. The Euler characteristic is independent of the particular triangulation you choose.

2. It is also invariant under bending or stretching the surface!

Further



$\chi = 2$

$\chi = 0$

$\chi = -2$

$\ldots \quad \chi = 2 - 2g$

g = # of holes or handles

Global Gauss-Bonnet Theorem.

If S is a compact orientable surface without boundary, then

$$\iint_S K \, dArea = 2\pi \chi_P(S),$$

for any triangulation P of S.

Proof. We know each triangle adds 3 edges, but each edge is shared by two triangles. So

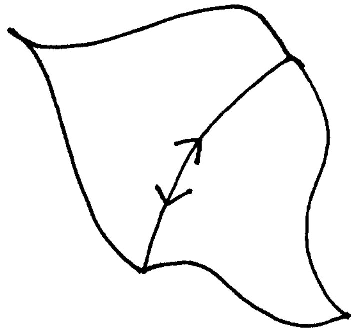
$$3F = 2E.$$

By Gauss-Bonnet,

$$\iint_S K \, dArea = \sum_{\Delta_i \in P} \iint_{\Delta_i} K \, dArea$$

$$= \sum_{\Delta_i \in P} 2\pi - \sum_{j=1}^{3} \epsilon_{ji} - \sum_{j=1}^{3} \int_{\alpha_{ji}} K_g(s) \, ds.$$

Along each edge α_{ji}, the geodesic curvature integral appears twice with opposite signs.



At each vertex, write $\epsilon_{ji} = \pi - \theta_{ji}$ where θ_{ji} is the interior angle. Then we can write

$$2\pi - \sum \epsilon_{ji} = \sum \theta_{ji} - \pi$$

On the other hand all the angles at each vertex sum to 2π, so rearranging the sums,

$$\begin{aligned}\iint K \, dArea &= 2\pi V - \pi F \\ &= 2\pi V - \pi F + \pi(3F - 2E) \\ &= 2\pi (V - E + F) \\ &= 2\pi \chi_P(S).\end{aligned}$$

Notice that we just proved amazing fact 1: we got the same lhs regardless of triangulation.

A slightly more complicated but similar argument (see Do Carmo) shows

Global Gauss-Bonnet. If S is a compact orientable surface with boundary,

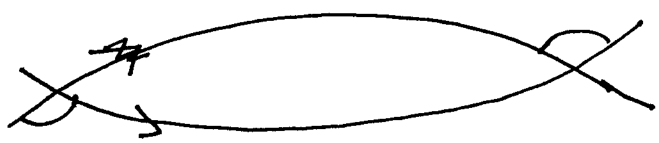
$$\int_{\partial S} K_g(s)\, ds + \iint_S K \, dArea + \sum \epsilon_i = 2\pi \chi(S)$$

where the ϵ_i are any exterior angles at corners of ∂S.

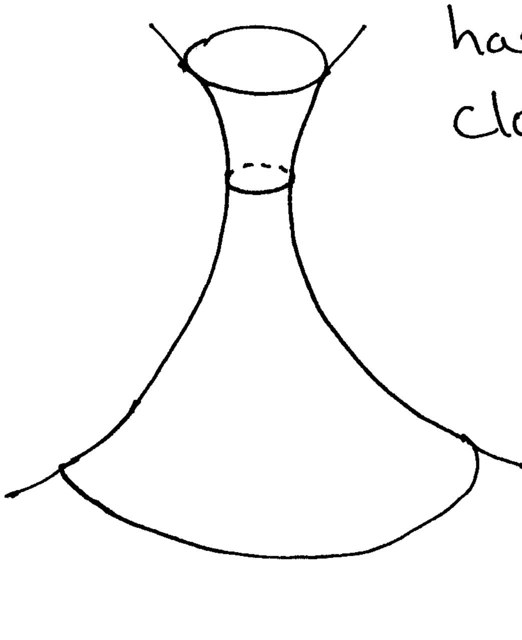
Consequences.

1. A compact surface of positive curvature is (topologically) a sphere.

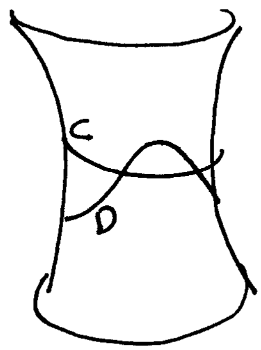
2. Two geodesic rays on a surface of negative curvature coming from a point can never meet again (in such a way that they bound a simply connected region)



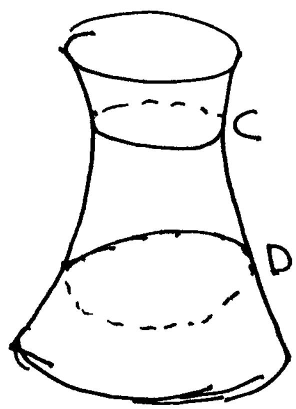
3. Let S be a (topological) cylinder with negative curvature. Then S has <u>at most one</u> simple closed geodesic.



Proof. Suppose we had two such curves C and D. If C, D intersect then "consecutive" intersections bound a simply connected region. $\times\!\!\times$



So we have



.

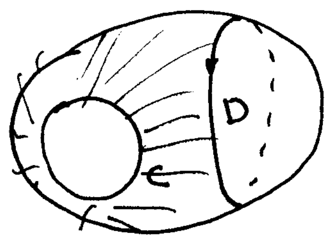
But then the region between C and D has $\chi = 0$ so

$$\iint_R K \, dArea = 0.$$

But $\iint_R K \, dArea < 0$. $\times\!\!\times$.

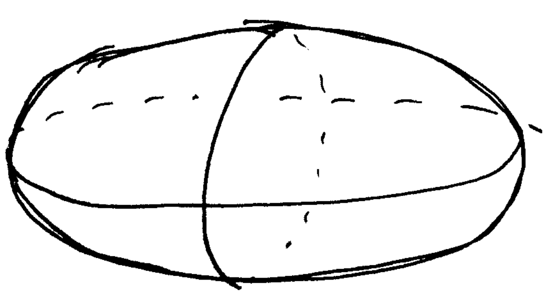
4. Any two simple closed geodesics on a surface of positive curvature intersect as well. ~~Not the sam~~

By 1, the surface is a sphere, so



the region between C and D has $\chi = 0$. Then arg't. is the same as before.

We're now done with our introduction to differential geometry. There are amazing vistas yet to explore!



Example. On any topological sphere in $\mathbb{R}^3$, there are at least 3 simple closed geodesics.