

Curves of constant curvature and torsion

We have now defined curvature and torsion for arbitrary parametrizations of a curve. Of course, our description isn't all that useful, since we must reparametrize by arclength to compute the values of κ and τ .

We now present formulae for $\kappa(t)$ and $\tau(t)$.

$$\kappa(t) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}$$

$$\tau(t) = \frac{-(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{|\gamma'(t) \times \gamma''(t)|^2}$$

Proving these is a homework assignment, but here's a hint:

$$\frac{d}{dt} \gamma(s(t)) = \gamma'(s) \cdot s'(t)$$

where $s'(t) = |\dot{\gamma}(t)|$, as we saw a moment ago at the end of lecture 2. Thus

$$\gamma'(s) = \frac{\gamma'(t)}{|\gamma'(t)|}$$

Thus we have

$$\begin{aligned}\frac{d^2}{dt^2} \gamma''(s(t)) &= \frac{d}{dt} (\gamma'(s) \cdot s'(t)) \\ &= \gamma''(s) \cdot (s'(t))^2 + \gamma'(s) \cdot s''(t)\end{aligned}$$

Here $s'(t) = |\gamma'(t)|$ so

$$s''(t) = \frac{1}{2} \langle \gamma'(t), \gamma'(t) \rangle^{-1/2} \cdot 2 \langle \gamma'(t), \gamma''(t) \rangle$$

or

$$s''(t) = \frac{\langle \gamma'(t), \gamma''(t) \rangle}{|\gamma'(t)|^2}$$

I leave it to you to solve for $\gamma''(s)$ and find its norm from the work so far.

Example. A curve of constant curvature and torsion is a helix.

(In fact, the converse is also true, but we leave that for you to prove.)

Suppose $x(s) \equiv x$ and $y(s) \equiv y$. We want to prove that γ is a helix. Consider the curve

$$\alpha(s) = \gamma''(s) = xN(s).$$

We first prove that $\alpha(s)$ is a circle.

1. $\alpha(s)$ has constant norm x , so α is on a sphere of radius x around the origin.
2. We claim α is planar.

We compute

$$\begin{aligned}\alpha'(s) &= x(-xT - yB) \\ &= -x^2T - xyB\end{aligned}$$

$$\begin{aligned}\alpha''(s) &= -x^3N - xy^2N \\ &= -x(x^2 + y^2)N\end{aligned}$$

$$\begin{aligned}\alpha'''(s) &= +x^2(x^2 + y^2)T + xy(x^2 + y^2)B \\ &= -(x^2 + y^2)(-x^2T - xyB) \\ &= -(x^2 + y^2)\alpha'(s).\end{aligned}$$

Thus the triple product in numerator of

$$\gamma_\alpha(s) = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

must vanish, since α''' is a scalar multiple of α' . We then see that $\gamma_\alpha(s) \equiv 0$, so α is planar.

Since $\alpha(s)$ is a circle of radius K , and $\alpha'(s)$ has norm $K\sqrt{x^2+y^2}$, we can write

$$\gamma''(s) = \alpha''(s) = (K \cos \sqrt{x^2+y^2} s, K \sin \sqrt{x^2+y^2} s, \overset{0}{\cancel{K}})$$

~~where $\gamma''(s) \neq 0$~~

Integrating twice with respect to s , we get

$$\gamma(s) = \left(\frac{-K}{x^2+y^2} \cos \sqrt{x^2+y^2} s, \frac{-K}{x^2+y^2} \sin \sqrt{x^2+y^2} s, Cs + D \right)$$

We can take $D=0$ by choosing coordinates so that $\gamma(0) = \left(\frac{-K}{x^2+y^2}, 0, 0 \right)$. To solve for C , we observe that

$$1 = |\gamma'(s)|^2 = \frac{K^2}{x^2+y^2} + \frac{K^2}{x^2+y^2} + C^2, \text{ or } C^2 = \gamma^2 - K^2.$$

With this example in hand, it may seem that, constant κ + constant τ is very restrictive, since

we should get some restrictions from assuming that

$$\kappa \equiv C \quad \text{or} \quad \tau \equiv C.$$

Not so!

Theorem [Ghomi, 2006] If γ is a curve of maximum curvature K and $K_2 \geq K$ then there is a curve γ_2 of constant curvature K_2 for any given $\epsilon > 0$ so that

$$|\gamma - \gamma_2| < \epsilon \quad \text{and} \quad |\gamma' - \gamma_2'| < \epsilon.$$

A similar statement holds for curves of constant torsion. Further, both theorems provide closed approximations to closed initial curves γ .

