Regular Surfaces

We defined curves as mappings $\mathbb{R} \to \mathbb{R}^3$, and then dealt with some ambiguity: many different parametrizations can have the same image.

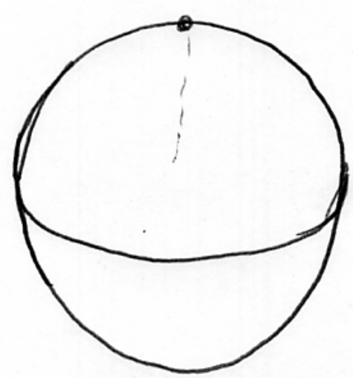
This creates a problem – are our geometric invariants functions of <u>images</u>? Or <u>parametrizations</u>?

We solved the problem with the <u>reparametrization theorem</u>, which showed (among other things) that curvature and torsion are the same for any <u>regular</u> parametrization of a curve.

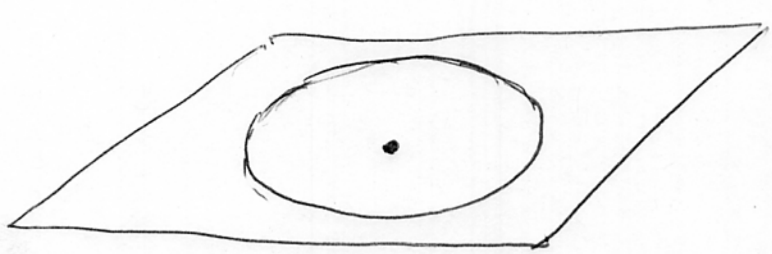
A missing page gave
the defn of regular
surface

Examples of regular surfaces:

We first claim $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ is a regular surface. We first construct a system of local coordinates around the North pole $(0,0,1)$.



$$\mathbf{x}(x,y) = (x, y, +\sqrt{1-x^2-y^2})$$



Clearly, $\mathbf{x}$ is differentiable on the open disk $|(x,y)| < 1$.

Let's compute Dx:

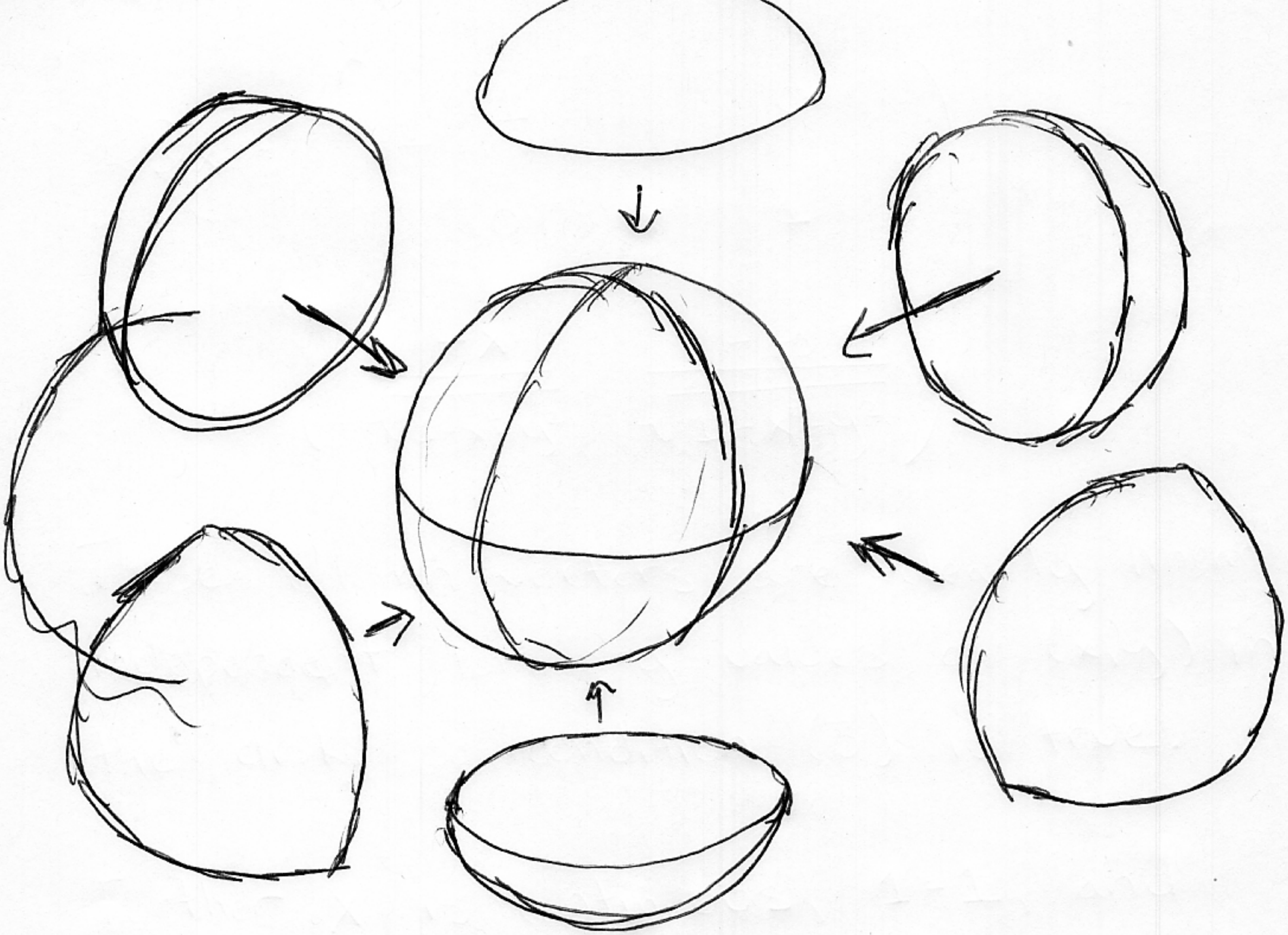
$$Dx = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix}.$$

These column vectors are clearly linearly independent for any values of (x,y) in the disk, so condition (3) is true.

Further x is continous, 1-1, and the inverse $x^{-1}(x,y,z) = (x,y)$ is continuous as well.

This system of local coordinates will work for any point in the (open) upper hemisphere.

To cover the entire sphere, we need

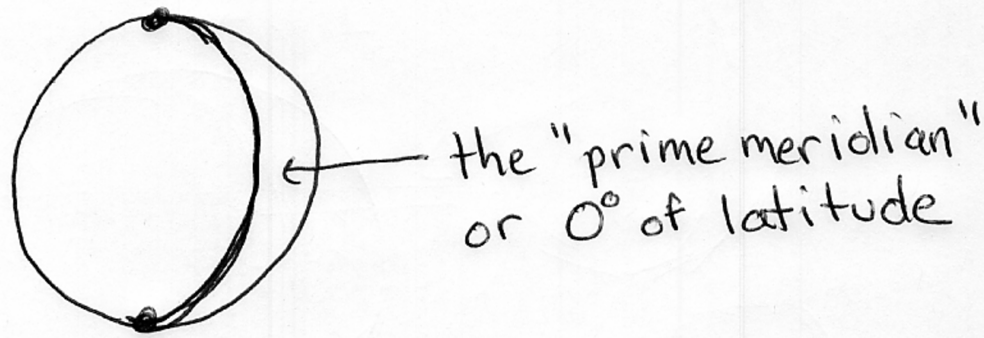


six of these parametrizations!
But that suffices to cover the sphere.

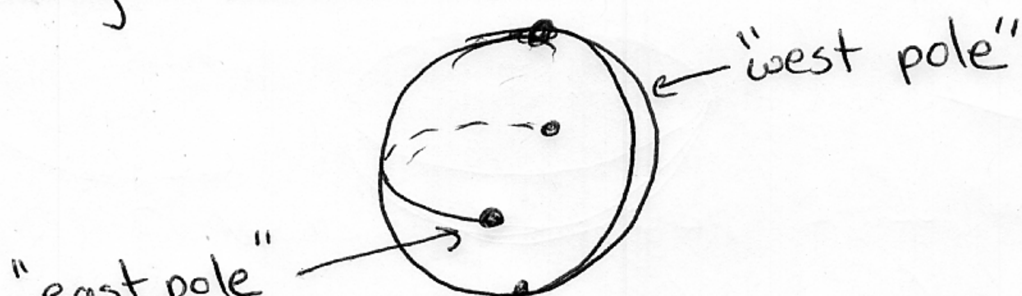
Another interesting example is the spherical coordinates θ, φ on S^2 given by

$$(\theta, \varphi) \longmapsto (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

It turns out that these coordinates are a parametrization everywhere <u>except</u> on a semicircle joining the poles



So we could parametrize S^2 with only <u>two</u> of these maps:



It's a giant pain to check for local parametrizations explictly: can we shortcut this process?

Sometimes...

Here's a classical example of a regular surface:

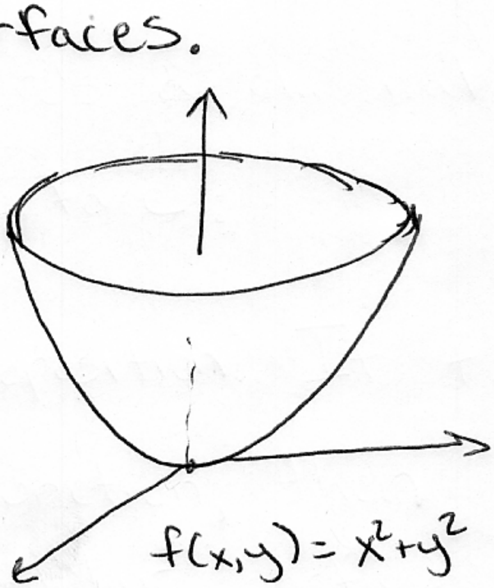
Proposition. If $f: U \to \mathbb{R}$ ($U \subset \mathbb{R}^2$) is a differentiable function, then the graph of f is a regular surface.

Proof. We parametrize by $(x,y) \overset{\vec{X}}{\mapsto} (x,y,f(x,y))$. This is continuous, 1-1, and onto, and has a continuous inverse $(x,y,f(x,y)) \mapsto (x,y)$. It is also differentiable. We need only check

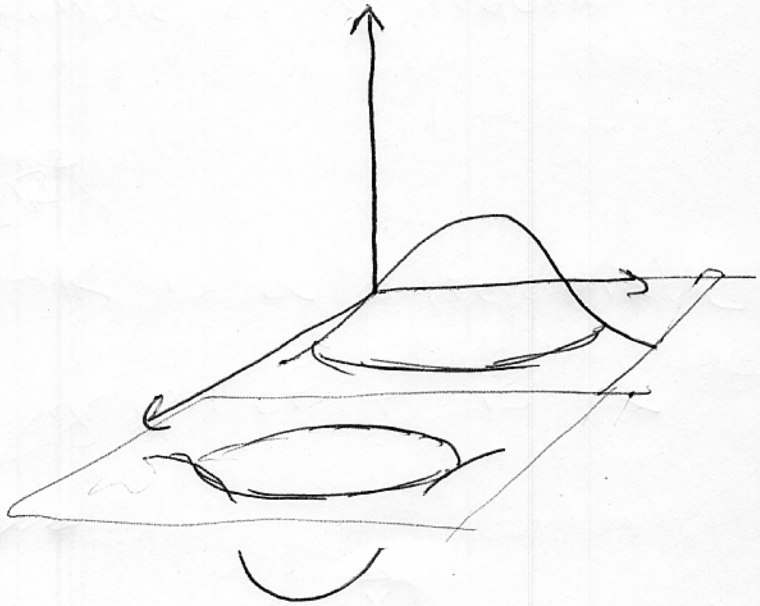
$$D\vec{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial \vec{X}}{\partial x} & \frac{\partial \vec{X}}{\partial y} \end{bmatrix}$$

and these column vectors are linearly independent. $\therefore$

Thus, graphs provide a class of regular surfaces.



$f(x,y) = x^2+y^2$



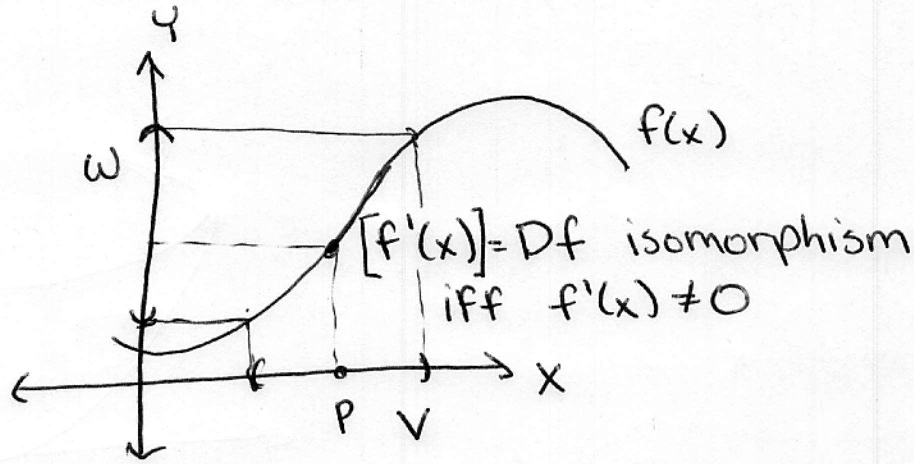
$f(x,y) = \sin(x+y)$.

In fact, <u>every</u> regular surface is locally a graph! To prove it, we'll need to recall

~~Proposition.~~

<u>The Inverse Function Theorem</u>. Let $F: U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. Suppose that at $\vec{p}$, the differential $DF_p : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Then $\exists$ neighborhoods V of $\vec{p}$ and W of $f(\vec{p})$ so that $F^{-1}: W \to V$ is a well-defined differentiable function.

The example to keep in mind is:



Now we have:

Proposition. Let $S \subset \mathbb{R}^3$ be a regular surface and $\vec{p} \in S$. Then $\exists$ a neighborhood V of $\vec{p}$ in S so that V is the graph of a differentiable function in one of the forms

$$z = f(x,y)$$
$$y = g(x,z)$$
$$x = h(y,z).$$

So our sphere example was really telling!

Proof. Let $\vec{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of S near $\vec{p}$. We can write

$$\vec{x}(u,v) = \big(x(u,v), y(u,v), z(u,v) \big).$$

By assumption (3)!, the vectors

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \text{ and } \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

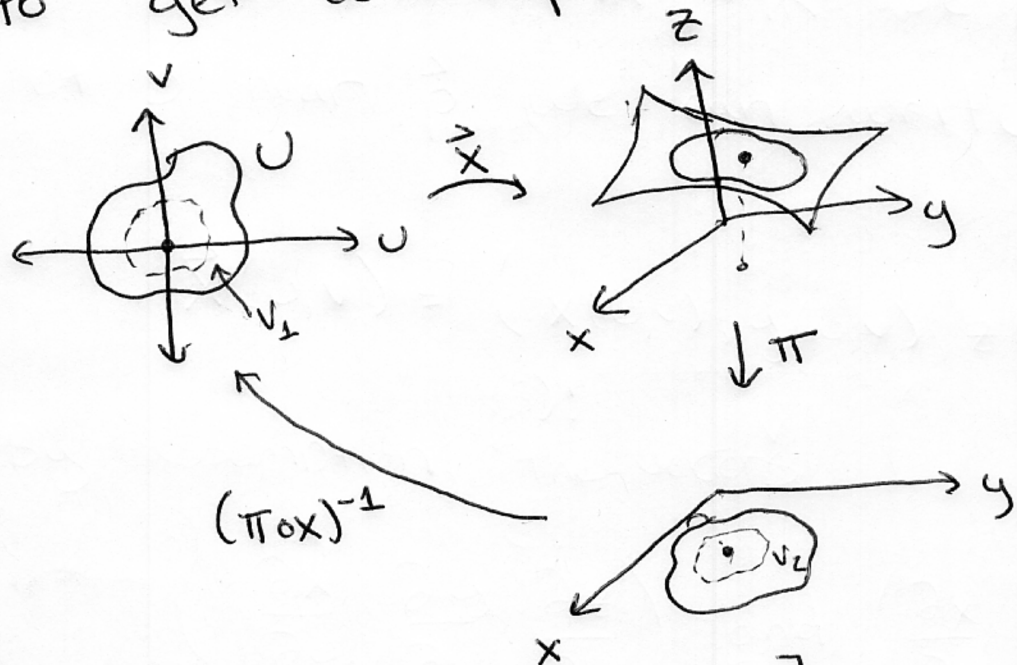
are linearly independant. Thus one of the coordinates of

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

is nonzero. Let's assume it's the last one,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0.$$

We can now project from the image of $\vec{x}$ to the x-y plane to get a map $\pi \circ \vec{x} : U \to \mathbb{R}^2$



Now $D(\pi \circ \vec{x}) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ (surprise!)

so by our assumption this is a nonsingular matrix (and hence an <u>isomorphism</u>).

Thus by the inverse function theorem, $\exists$ some V_2 around $\pi(\vec{p})$ and $V_1 \subset U$ so that there is a differentiable inverse map

$$(\pi \circ x)^{-1} : V_2 \to V_1.$$

Now if we compose

$$z \circ (\pi \circ x)^{-1} = z\left((\pi \circ x)^{-1}(x,y)\right),$$

we claim that S is locally the graph

$$(x, y, z((\pi \circ x)^{-1}(x,y))).$$

To check this, we need to prove that any such point is actually <u>on</u> the surface S! ~~For any (x,y),~~

$$\cancel{\pi \circ \vec{X} \circ (\pi \circ \vec{X})^{-1}(x,y) = (x,y)}$$

~~Thus~~

$$\cancel{\vec{X} \circ (\pi \circ \vec{X})}$$

For any $(x,y) \in V_2$, $\vec{X} \circ (\pi \circ \vec{X})^{-1}(x,y)$ is certainly on S. We can write <u>that</u> point as

$$\left(x\left((\pi \circ \vec{X})^{-1}(x,y)\right), y\left((\pi \circ \vec{X})^{-1}(x,y)\right), z\left((\pi \circ \vec{X})^{-1}(x,y)\right) \right)$$

Of course, the projection π of this point has to be (x,y) again. Thus

$$x\left((\pi\circ\vec{x})^{-1}(x,y)\right) = x$$

and

$$y\left((\pi\circ\vec{x})^{-1}(x,y)\right) = y$$

and $\left(x, y, z\left((\pi\circ x)^{-1}(x,y)\right)\right)$ is on S, as we desired! $\therefore$

We will often use this description of regular surfaces in the proofs to come.

Another great source of regular surfaces comes from level surfaces of scalar functions on $\mathbb{R}^3$.